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A Novel Method for Simplified Reliability-based Code Calibration

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ABSTRACT

This paper compares two methods for geotechnical reliability code calibration, the well known design value method (DVM) based on first-order reliability method and a recently developed method based on quantile, called the quantile value method (QVM). The unique issue of non-constant coefficients of variation (COV) pertaining to geotechnical designs is studied in this paper. With the non-constant COVs, the main focus is to verify which method after being calibrated by using a single calibration case can provide more uniform reliability levels over a wide range of validation cases. A very simple geotechnical design example for which analytical solution is available is taken as the demonstrative example for the comparison. The results show that QVM is more robust than DVM in terms of providing uniform reliability level.

1. INTRODUCTION

The coefficients of variation (COV) for geotechnical parameters and model factors are not constant. Taking the undrained shear strength (s_u) of a clay as an example, measurement errors for undrained shear strengths obtained from unconfined compression (UC) tests are typically larger, compared to s_u obtained from more sophisticated tests such as isotropically consolidated undrained compression (CIUC) tests. Spatial averaging over a large volume of soil mass may also significantly reduce the COV in s_u (Vanmarcke 1977). In addition, there are various methods of estimating s_u. For example, s_u can be estimated from the preconsolidation stress or from the liquidity index. Different transformation equations are needed to convert the measured parameter (preconsolidation stress or liquidity index) to the desired design parameter (s_u). The transformation uncertainties can vary significantly as well (Phoon 1995). For example, the s_u versus preconsolidation stress transformation usually is associated with less transformation uncertainty than the s_u versus liquidity index transformation.

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Geotechnical models, such as the classical limit equilibrium models for pile capacity, are not exact. Model factors are needed to relate somewhat idealized calculations with measurements. It is well established that model factors are random variables, typically lognormally distributed. The mean and COV of a model factor are typically obtained from calibration with field measurements (e.g., pile load test database). These statistics may change depending on the database, even for the same problem and the same calculation model.

The issue of COVs varying over a wide range is quite often encountered in geotechnical design practice, because soil is a natural material and there is a diversity of testing methodologies developed to suit different site conditions. In contrast, concrete and steel are manufactured and testing methodologies are accordingly more standardized. Hence, structural design practice does not need to contend with COVs varying over a wide range. This issue must be dealt with in geotechnical reliability-based design, although it poses a significant challenge. To elaborate on this challenge, consider a simple pile design problem involving two variables, the resistance Q and the load L. Let V_Q and V_L be the COVs of the resistance and load, respectively. Assume that $V_L = 0.15$ is constant, but V_Q is not constant. Let scenario A be a case where a detailed site investigation and extensive load tests have been conducted. As a result, V_Q is small and equal to 0.2. Scenario B is a case where the site investigation is cursory and no load test is conducted. As a result, V_Q is large and equal to 0.45. It is evident that a set of constant load and resistance factors (or partial factors) cannot maintain a uniform reliability level over these two disparate scenarios. The challenge is obviously non-existent if one adopts a full probabilistic approach, rather than a simplified reliability-based design approach. It is assumed in this paper that the former is not acceptable to practitioners at the present moment, which is indeed the case in the geotechnical engineering community.

In this paper, a method named “quantile value method (QVM)” for calibrating partial factors is presented. This method is based on the quantile-based theoretical approach developed in Ching and Phoon (2011). The name QVM is herein selected to differentiate from the more widely known “design value method (DVM)” (Ditlevsen and Madsen 1996; Honjo et al. 2002) based on the first-order reliability method (FORM) (Hasofer and Lind 1974). Both DVM and QVM adopt conservative design locations situated on the limit state line, but DVM adopts the FORM design point, while QVM adopts a design location that was never explored in literature. In this study, DVM and QVM will be compared using a simple geotechnical pile design involving only two random variables. For this simple example, exact solutions for both DVM and QVM are available, so the comparison can be made analytically and geometric interpretations can be presented visually in the standard Gaussian space. The comparison focuses on the ability to maintain a uniform reliability level over a wide range of validation design scenarios, such as different COVs, using a single prescribed number (resistance factor or quantile). The analytical comparison will be mostly limited to the case where the prescribed number is calibrated from a single design scenario, but validation would cover a number of design scenarios. Calibration involving multiple design scenarios will be addressed numerically in association with two realistic geotechnical design examples. It will be shown that most of the issues encountered for the realistic examples can be explained by the theoretical insights garnered in the simple analytical example.

2. ANALYTICAL EXAMPLE

The following simple example is adopted to compare DVM and QVM analytically. Consider a pile with axial resistance Q and subjected to axial load L. Q and L are Coindependent and lognormally distributed with mean values (μ_Q, μ_L) and COVs (V_Q, V_L). The limit state function is defined to be G = ln(Q) – ln(L). In the standard Gaussian space,

$$g\left(z_Q, z_L\right) = \lambda_Q + \xi_Q z_Q - \lambda_L - \xi_L z_L \qquad \lambda = \ln\left(\mu \big/ \sqrt{1+V^2}\right) \qquad \xi = \sqrt{\ln\left(1+V^2\right)} \tag{1}$$

where λ and ξ are respectively the mean and standard deviation of the logarithm of the subscripted variable, and (z_Q, z_L) are jointly standard Gaussian. The safety ratio can be defined as

$$SR\left(z_Q, z_L\right) = \frac{Q}{L} = \exp\left(\lambda_Q + \xi_Q z_Q - \lambda_L - \xi_L z_L\right) \tag{2}$$

Whenever SR(z_Q, z_L) < 1, failure occurs, and vice versa.

Two cases would be considered: a calibration case and a validation case. The mean values and COVs for the calibration case are (μ_Q, μ_L) and COVs (V_Q, V_L), and those for the validation case are (μ'_Q, μ'_L) and COVs (V'_Q, V'_L). Basically, the calibration case will be used to calibrate the partial factors (or load and resistance factors) to achieve a prescribed target reliability index of β_T. The validation case will be used to examine whether these partial factors indeed produce a design with an actual reliability index β'_A that is reasonably close to β_T.

The geometric interpretation is illustrated in Fig. 1. For this simple example, the limit state lines are linear in the standard Gaussian space. However, the limit state lines for the calibration case (g = 0) and validation case (g' = 0) are different and are in general not parallel to each other. The reason is that $V_Q \neq V'_Q$ and $V_L \neq V'_L$.

3. DESIGN VALUE METHOD AND QUANTILE VALUE METHOD

Reliability-based design is typically implemented in design codes using a set of partial factors (or load and resistance factors) that achieves the target reliability index β_T for the calibration case. However, this set of numbers is not unique. In the standard Gaussian space, any point on the "adjusted" limit state line $g(z) = \lambda_Q + \xi_Q z_Q - \lambda_L - \xi_L z_L = 0$ can be used to derive a set of partial factors. The adjusted limit state line is a limit state line with distance to the origin adjusted to β_T. The chosen point on the adjusted limit state line for evaluation of partial factors will be called the “design location” in this paper. The design location is not necessarily the same as the widely known FORM “design point”: the design location can be anywhere on the limit state line g = 0, and the FORM design point is only a special case. It is the point on g = 0 nearest to the origin.

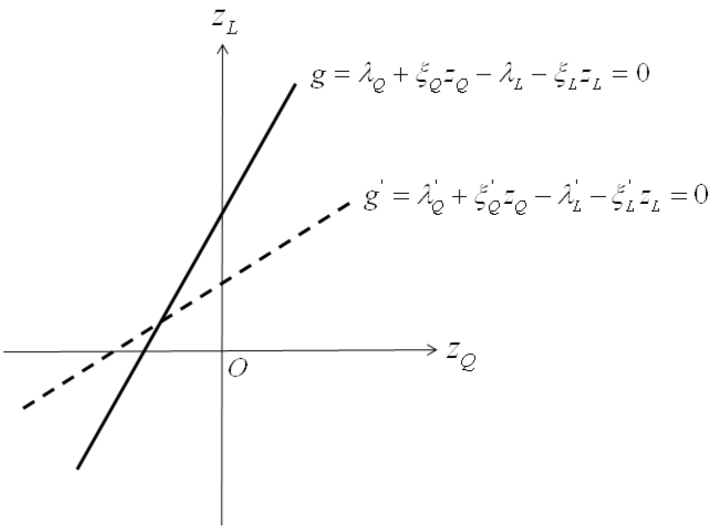


Fig. 1 Geometric interpretations for the limit state lines of the calibration case (g = 0) and validation case (g’ = 0).

Before choosing a design location on the limit state line g = 0 for the calibration case, the limit state line must be adjusted to a distance of β_T from the origin to fulfill the target reliability index. This can be done by adjusting one or several of the design parameters $\{\mu_Q, \mu_L, V_Q, V_L\}$ or, equivalently, among $\{\lambda_Q, \lambda_L, \xi_Q, \xi_L\}$. It is usually impractical to adjust the COVs. The mean resistance can be increased by lengthening the pile and the mean load can be reduced by distributing the column load to multiple piles in a pile group. In the ensuing analysis, λ_L is adjusted to a value equal to $\lambda_Q - \beta_T(\xi_Q^2+\xi_L^2)^{0.5}$. The adjusted design parameter will be called the “pivoting design parameter” in this paper. After the adjustment, the limit state line becomes

$$g(z) = \xi_Q z_Q - \xi_L z_L + \beta_T \sqrt{\xi_Q^2 + \xi_L^2} = 0 \tag{3}$$

Note that this adjustment is carried out on the calibration case, not on the validation case. There are many possible choices for the set of partial factors for the calibration case. The DVM and QVM are two special cases. Both methods select design locations on the adjusted limit state line but impose some restrictions explained below so that the resulting set is unique.

3.1 Design value method

The design value method (DVM) chooses the design location to be the FORM design point, which is the point on the adjusted limit state line that is closest to the origin (shown as z^* in the left plot of Fig. 2). It is also the most probable point on the adjusted limit state line for the calibration case. Direct calculation shows that the FORM design point has the following coordinates:

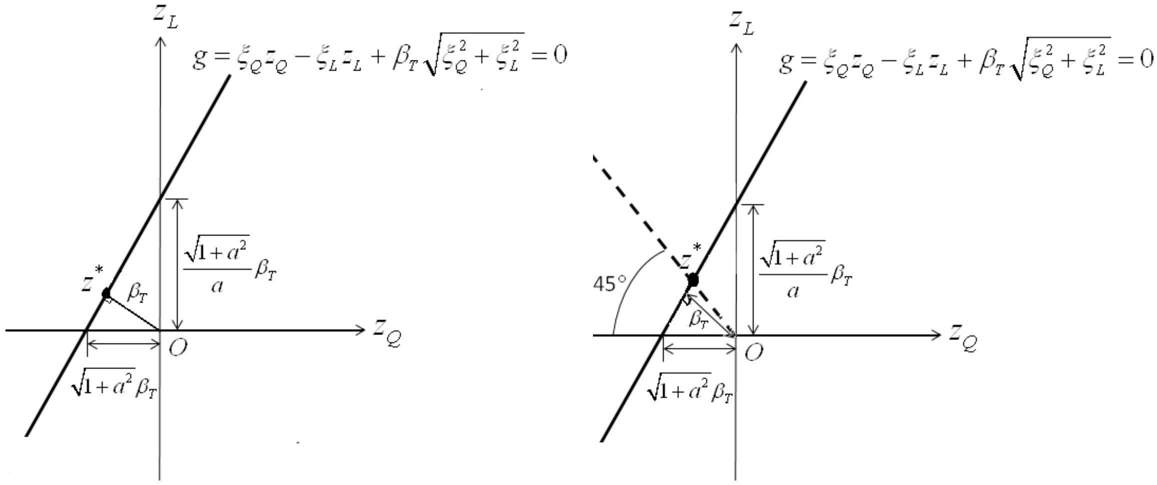


Fig. 2 Design locations z^* for the calibration case for DVM (left) and QVM (right) ($a = \xi_L/\xi_Q$). Both limit state lines are adjusted to a distance of β_T from the origin.

$$z_Q^* = \frac{-\beta_T \xi_Q}{\sqrt{\xi_Q^2 + \xi_L^2}} \qquad z_L^* = \frac{\beta_T \xi_L}{\sqrt{\xi_Q^2 + \xi_L^2}} \tag{4}$$

The corresponding Q^* and L^* are

$$Q^* = \exp\left(\lambda_Q - \beta_T \xi_Q^2 \Big/ \sqrt{\xi_Q^2 + \xi_L^2}\right) \qquad L^* = \exp\left(\lambda_L + \beta_T \xi_L^2 \Big/ \sqrt{\xi_Q^2 + \xi_L^2}\right) \tag{5}$$

By definition, the algebraic design equation, $Q^* - L^* = 0$, for the calibration case will be associated with a reliability index identical to β_T. This design equation can be expressed in the alternate form, $\gamma_Q\mu_Q - \gamma_L\mu_L = 0$, where ($\gamma_Q$, γ_L) are the resistance and load factors calibrated from the calibration case

$$\gamma_Q = \exp\left(-0.5\xi_Q^2 - \beta_T \xi_Q^2 \Big/ \sqrt{\xi_Q^2 + \xi_L^2}\right) \qquad \gamma_L = \exp\left(-0.5\xi_L^2 + \beta_T \xi_L^2 \Big/ \sqrt{\xi_Q^2 + \xi_L^2}\right) \tag{6}$$

The equation, $\gamma_Q\mu_Q - \gamma_L\mu_L = 0$, is also called the Load and Resistance Factor Design (LRFD) format. In this LRFD format, the nominal load and resistance are assumed to be equal to their respective mean values for simplicity. Although the LRFD format is exact for the calibration case, it is of limited practical interest as the purpose of recommending a design equation in a design code is to apply it over a range of common encountered design scenarios. It is rarely mentioned that the LRFD format is assumed to work adequately for other design scenarios without re-calibration of the resistance/load factors.

It is of interest to know the actual reliability index, denoted by β'_A, implied by these partial factors (exactly applicable only for the calibration case) when they are applied to the validation case. Although it is evident that $\beta'_A \neq \beta_T$, it is important to know the difference particularly for $\beta'_A < \beta_T$. The same design equation is applied to the validation case. The only difference is that the calibrated factors are applied to the mean resistance (μ'_Q) and mean load (μ'_L) for the validation case

$$\gamma_Q \mu'_Q = \gamma_L \mu'_L \qquad or \qquad \ln\left(\gamma_Q\right) + \ln\left(\mu'_Q\right) = \ln\left(\gamma_L\right) + \ln\left(\mu'_L\right) \tag{7}$$

Or equivalently,

$$-0.5\xi_Q^2 - \beta_T \xi_Q^2 \Big/ \sqrt{\xi_Q^2 + \xi_L^2}\, \gamma_Q + \underbrace{\ln\left(\mu'_Q\right)}_{\lambda'_Q + 0.5\xi_Q'^2} = -0.5\xi_L^2 + \beta_T \xi_L^2 \Big/ \sqrt{\xi_Q^2 + \xi_L^2} + \underbrace{\ln\left(\mu'_L\right)}_{\lambda'_L + 0.5\xi_L'^2} \tag{8}$$

The actual reliability index β'_A for the validation case implied by the partial factors (γ_Q, γ_L) is

$$\beta'_A = \frac{\lambda'_Q - \lambda'_L}{\sqrt{\xi_Q'^2 + \xi_L'^2}} = \frac{-0.5\left(\xi_L^2 - \xi_L'^2\right) + 0.5\left(\xi_Q^2 - \xi_Q'^2\right) + \beta_T \sqrt{\xi_Q^2 + \xi_L^2}}{\sqrt{\xi_Q'^2 + \xi_L'^2}} \tag{9}$$

Note that β'_A does not depend on the mean values (μ_Q, μ_L) but only on the COVs (V_Q, V_L) in this example. It is evident from Eq. (9) that $\beta'_A = \beta_T$ if the validation case has exactly the same (V_Q, V_L) as the calibration case, i.e., $\xi'_Q = \xi_Q$ and $\xi'_L = \xi_L$. However, if the validation case has very different (V'_Q, V'_L), β'_A may be far from β_T. The departure is theoretically quantified in Eq. (9). In particular, it is possible for for $\beta'_A < \beta_T$.

Consider a scenario where $V_L = 0.2$ and $V_Q = 0.3$ for the calibration case, and consider $V'_L = V_L$ and V'_Q ranges from 0.1 to 0.5 for the validation case. The target reliability index is $\beta_T = 3.0$ (corresponding to a target failure probability p_T = 1.3e-3). The calibrated partial factors from Eq. (6) are $\gamma_Q = 0.462$ and $\gamma_L = 1.367$. This means that for the validation case, one needs to assure $0.462\mu'_Q = 1.367\mu'_L$ in the natural (or physical) space of Q and L. The actual reliability index β'_A under various V'_Q is plotted as the thick solid line in the left plot in Fig. 3. It is clear that β'_A may be as high as 5.0 (actual failure probability p'_A = 2.9e-7) when $V'_Q = 0.1$ and as low as 2.0 (p'_A = 2.3e-2) when $V'_Q = 0.5$. The uniform reliability level is not achieved by the calibrated partial factors $\gamma_Q = 0.462$ and $\gamma_L = 1.367$. In particular, unconservative designs could be produced.

3.2 Quantile value method

The basic idea of the quantile value method (QVM) is to reduce the resistance Q to its η quantile (η is small) but to increase the load L to its 1-η quantile. The parameter η is called the probability threshold, and the same threshold η is applied to both random variables: taking η quantiles for stabilizing variables and 1-η quantiles for destabilizing variables. In the standard Gaussian space, this is equivalent to selecting the design location z^* to be a point on the adjusted limit state line for the calibration case that satisfies $z_Q = -z_L = \Phi^{-1}(\eta)$ (shown as z^* in the right plot of Fig. 2). Ching and Phoon (2011) showed that the relation between η and β_T is as follows:

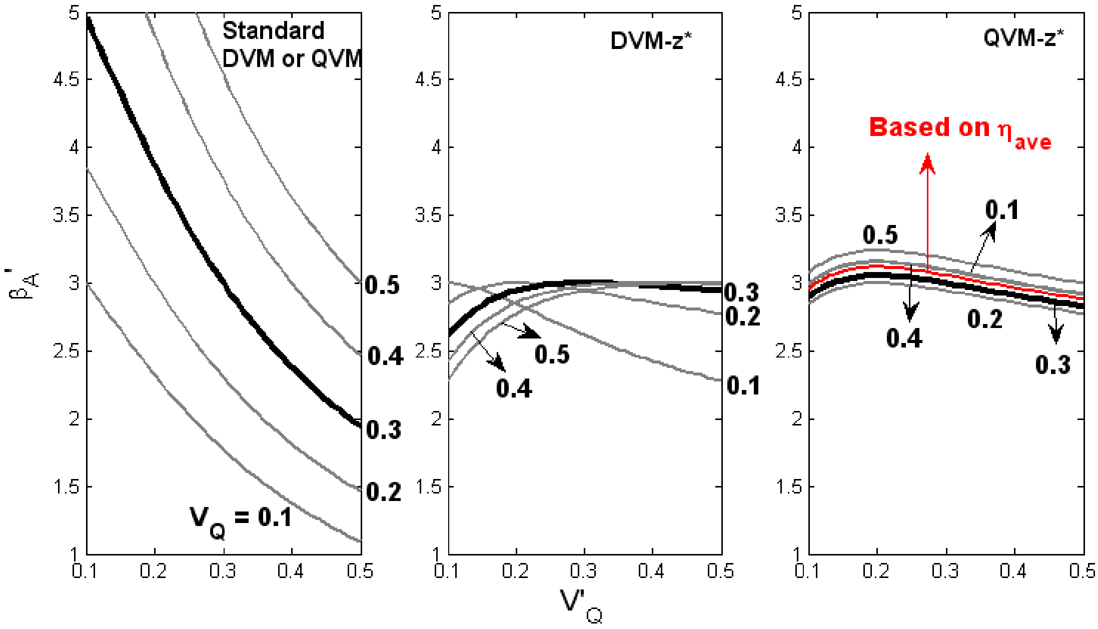


Fig. 3 The actual reliability index β'_A for the validation case.

$$P\left(\frac{SR\left[z_Q=\Phi^{-1}(\eta),z_L=-\Phi^{-1}(\eta)\right]}{SR\left(z_Q,z_L\right)}>1\right)=\Phi\left(-\beta_T\right) \tag{10}$$

This leads to

$$P\left(\frac{\exp\left[\xi_Q\Phi^{-1}(\eta)+\xi_L\Phi^{-1}(\eta)+\beta_T\sqrt{\xi_Q^2+\xi_L^2}\right]}{\exp\left(\xi_Q z_Q-\xi_L z_L+\beta_T\sqrt{\xi_Q^2+\xi_L^2}\right)}>1\right)=\Phi\left(\frac{\left(\xi_Q+\xi_L\right)\Phi^{-1}(\eta)}{\sqrt{\xi_Q^2+\xi_L^2}}\right)=\Phi\left(-\beta_T\right) \tag{11}$$

As a result, the calibrated η is

$$\eta = \Phi\left(\frac{-\beta_T\sqrt{\xi_Q^2+\xi_L^2}}{\xi_Q+\xi_L}\right) \tag{12}$$

Or equivalently, the design location has the following coordinates in standard normal space:

$$z_Q^* = \Phi^{-1}(\eta) = \frac{-\beta_T\sqrt{\xi_Q^2+\xi_L^2}}{\xi_Q+\xi_L} \qquad z_L^* = -\Phi^{-1}(\eta) = \frac{\beta_T\sqrt{\xi_Q^2+\xi_L^2}}{\xi_Q+\xi_L} \tag{13}$$

The corresponding Q^* and L^* are

$$Q^* = \exp\left(\lambda_Q - \beta_T\xi_Q\sqrt{\xi_Q^2+\xi_L^2}\Big/\left(\xi_Q+\xi_L\right)\right) \qquad L^* = \exp\left(\lambda_L + \beta_T\xi_L\sqrt{\xi_Q^2+\xi_L^2}\Big/\left(\xi_Q+\xi_L\right)\right) \tag{14}$$

and the corresponding resistance and load factors are

$$\gamma_Q = \exp\left(-0.5\xi_Q^2 - \beta_T\xi_Q\sqrt{\xi_Q^2+\xi_L^2}\Big/\left(\xi_Q+\xi_L\right)\right) \qquad \gamma_L = \exp\left(-0.5\xi_L^2 + \beta_T\xi_L\sqrt{\xi_Q^2+\xi_L^2}\Big/\left(\xi_Q+\xi_L\right)\right) \tag{15}$$

If these partial factors (originally calibrated for the calibration case) are applied to the validation case, the actual reliability index can be derived based the steps similar to Eqs. (7)-(9). Quite surprisingly, the actual reliability index β'_A for the validation case implied by the calibrated partial factors (γ_Q, γ_L) has the same expression as Eq. (9), i.e. exactly the same as the DVM result, although the partial factors for QVM are different from those for DVM. As a result, a uniform reliability level cannot be achieved by QVM, either.

4. STRATEGIES OF VARIABLE PARTIAL FACTORS

Note that the above DVM and QVM results are based on the strategy of applying partial factors calibrated for the calibration case to the validation case. The underlying assumption is that there may be a set of constant partial factors that are suitable for both design cases. Intuitively, this is unlikely to be true because partial factors should be closely related to the uncertainty level, or COVs – the random variables with large COVs should be multiplied by partial factors far away from 1, and vice versa. It has been observed in the Introduction that the COVs for structural materials only vary in a narrow range and it is sensible to apply constant partial factors for structural reliability-based design. On the other hand, the COVs of geo-materials vary over a wide range. There is an emerging realization in the geotechnical reliability literature that the direct application of LRFD or similar approaches with constant partial factors is overly simplistic. In this section, strategies involving variable partial factors based on DVM and QVM are studied.

4.1 DVM with variable partial factors

In DVM, instead of applying partial factors, it may be possible to apply the FORM design point computed from the calibration case. In other words, we assume that the design point for the calibration case is applicable to the validation case, rather than assuming that the partial factors for the calibration case is applicable to the validation case. The motivation for this assumption is that the design point is defined in standard normal space and it is hopefully less sensitive to COVs. This approach of applying the FORM design point is denoted by "DVM-z^*", and the standard approach of applying the partial factors is denoted by "DVM". It will be clarified later that DVM-z^* has certain advantages in terms of maintaining a uniform reliability level. For standard DVM, one needs to assure that the design constraint $\gamma_Q\mu'_Q = \gamma_L\mu'_L$ is fulfilled for the validation case – this is done in the natural space of Q and L. For DVM-z^*, one needs to assure the FORM design point for the calibration case is also on the limit state line for the validation case – this is done in the standard Gaussian space. For DVM-z^*, the limit state line for the validation case is forced to pass through the design point in Eq. (4), i.e.,

$$\lambda'_Q + \xi'_Q z^*_Q - \lambda'_L - \xi'_L z^*_L = \lambda'_Q - \lambda'_L - \beta_T \frac{\xi_Q \xi'_Q + \xi_L \xi'}{\sqrt{\xi_Q^2 + \xi_L^2}} = 0 \tag{16}$$

As a result, the actual reliability index β'_A for the validation case implied by the FORM design point (z_Q^*, z_L^*) is

$$\beta'_A = \frac{\lambda'_Q - \lambda'_L}{\sqrt{\xi'^2_Q + \xi'^2_L}} = \frac{\xi_Q \xi'_Q + \xi_L \xi'_L}{\sqrt{\xi_Q^2 + \xi_L^2} \times \sqrt{\xi'^2_Q + \xi'^2_L}} \beta_T = \frac{1 + a \times a'}{\sqrt{1 + a^2} \times \sqrt{1 + a'^2}} \beta_T \tag{17}$$

where the ratio $a = \xi_L/\xi_Q$ and $a' = \xi'_L/\xi'_Q$. Note that β'_A does not depend on the mean values (μ_Q, μ_L) but only on the COVs (V_Q, V_L) through the ratios (a, a'), and $\beta'_A = \beta_T$ if the validation case has exactly the same ratio as the calibration case, i.e., $\xi'_L/\xi'_Q = \xi_L/\xi_Q$. This is somewhat less strict than standard DVM, which requires $\xi'_Q = \xi_Q$ and $\xi'_L = \xi_L$ to achieve the prescribed target reliability index.

Although $\beta'_A = \beta_T$ can occur with a less strict condition for DVM-z^*, a closer examination of Eq. (17) reveals that β'_A cannot be greater than β_T. This implies that implementing the FORM design point calibrated for a target level β_T to another case will always lead to a design with β'_A no greater than β_T, i.e., the validation case is always unconservative. This issue will be referred to as the unconservative design issue for subsequent discussion. This is because $(1+a \times a')^2 \leq (1+a^2) \times (1+a'^2)$, following the Cauchy-Swartz Inequality (Steele 2004). This phenomenon can be explained geometrically in the left plot in Fig. 4. Recall that the adjusted limit state line for the calibration case has a distance exactly equal to β_T to the origin. For DVM-z^*, the calibrated design location for the calibration case is at the FORM design point (z^* in the left plot). Substituting this design location to the validation case is equivalent to enforcing the limit state line g' = 0 in the plot to pass through z^*. It is clear that the distance of from the limit sate line g' = 0 to the origin cannot be greater than β_T. This distance is exactly the actual reliability index β'_A for the validation case. Simple trigonometric calculations show that this distance is indeed $(1+a\times a')/(1+a^2)^{0.5}/(1+a'^2)^{0.5}\times\beta_T$, as derived in Eq. (17).

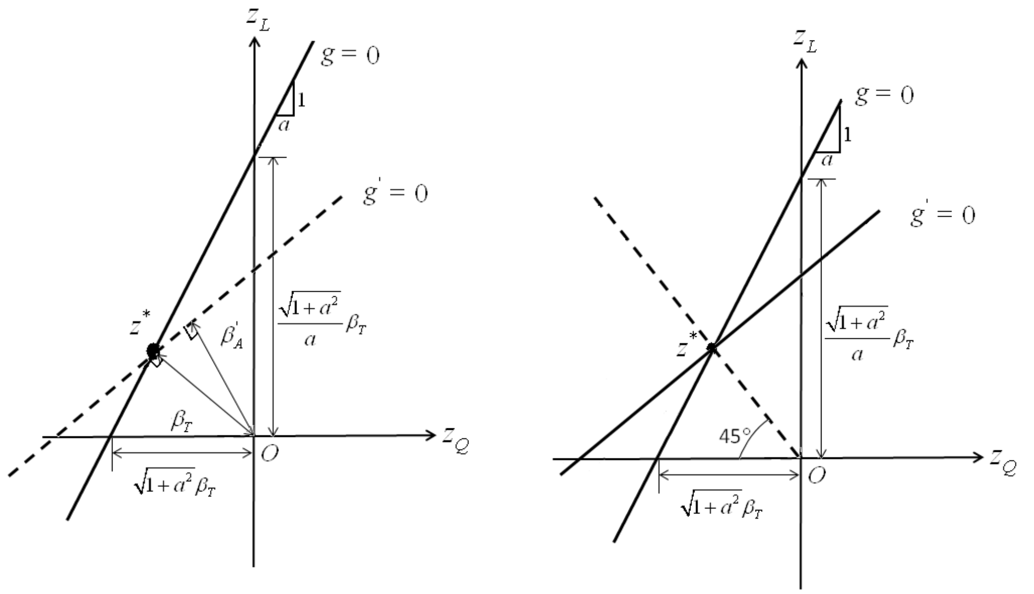


Fig. 4 Geometric interpretations for DVM-z^* (left) and for QVM-z^* (right).

Consider the same scenario where V_L = 0.2 and V_Q = 0.3 for the calibration case, and consider $V'_L = V_L$ and V'_Q ranges from 0.1 to 0.5 for the validation case, and the target is β_T = 3.0. The calibrated design point from Eq. (4) is (z^*_Q, γ^*_L) = (-2.487, 1.678). This means that for the validation case, one needs to comply with the algebraic design equation, $\lambda'_Q - \xi'_Q\times2.487 = \lambda'_L + \xi'_L\times1.678$ in the standard Gaussian space. Equivalently, one can state the design equation in the natural space of Q and L, $\gamma_Q\mu'_Q = \gamma_L\mu'_L$, where the partial factors (γ_Q, γ_L) now depend on the COVs for the validation case:

$$\gamma_Q = \exp\left(-0.5\xi_Q'^2 - 2.487\xi_Q'\right) \qquad \gamma_L = \exp\left(-0.5\xi_L'^2 + 1.678\xi_L'\right) \tag{18}$$

Hence, assuming that the design point is invariant implies variable partial factors. The resulting β'_A under various V'_Q is plotted as the thick solid line in the middle plot in Fig. 3. It is clear that β'_A is always less than or equal to 3.0. Compared to standard DVM or standard QVM results, β'_A from DVM-z^* is much more uniform, ranging from 2.6 for V'_Q = 0.1 to 2.95 for V'_Q = 0.5. Hence, DVM-z^* is more advantageous in terms of maintaining a uniform reliability level and assuming the FORM design point to be invariant seems to be a better than assuming the partial factors are invariant.

4.2 QVM with variable partial factors

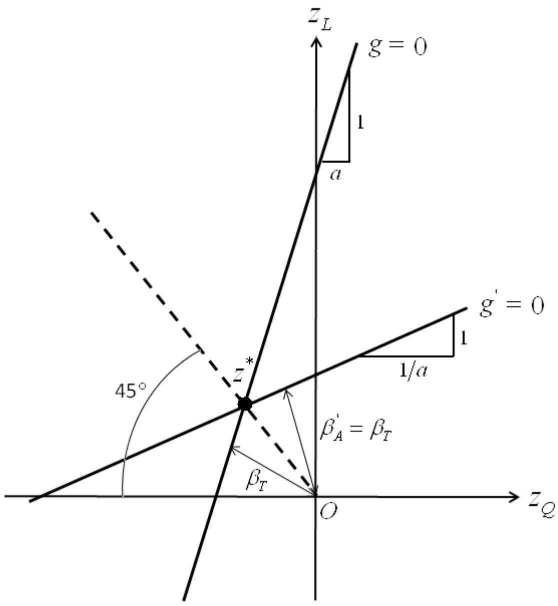
The strategy here is exactly the same as for DVM-z^*, except that the QVM design location is considered as invariant. Recall that the design location for QVM is not the FORM design point but a point on the 1-to-(-1) line in the standard Gaussian space, as seen in the right plot of Fig. 2. This approach of applying the QVM design location is denoted by "QVM-z^*". For QVM-z^*, one needs to assure the design location for the calibration case is also on the limit state line for the validation case – this is done in the standard Gaussian space. For QVM-z^*, the limit state line for the validation case is forced to pass through the design location in Eq. (13), i.e.,

$$\lambda'_Q + \xi'_Q z^*_Q - \lambda'_L - \xi'_L z^*_L = \lambda'_Q - \lambda'_L - \frac{\beta_T\left(\xi'_Q + \xi'_L\right)\sqrt{\xi_Q^2 + \xi_L^2}}{\xi_Q + \xi_L} = 0 \tag{19}$$

As a result, β'_A for the validation case implied by the design location (z_Q^*, z_L^*) is

$$\beta'_A = \frac{\lambda'_Q - \lambda'_L}{\sqrt{\xi'^2_Q + \xi'^2_L}} = \frac{\left(\xi'_Q + \xi'_L\right)\times\sqrt{\xi_Q^2 + \xi_L^2}}{\left(\xi_Q + \xi_L\right)\times\sqrt{\xi'^2_Q + \xi'^2_L}}\beta_T = \frac{1+a}{\sqrt{1+a^2}}\times\frac{\sqrt{1+a'^2}}{1+a'}\beta_T \tag{20}$$

The above β'_A can also be visualized geometrically as shown in the right plot in Fig. 4. Applying the QVM design location to the validation case is equivalent to enforcing the limit state line g' = 0 to pass through z^* located on the 1-to-(-1) line. The distance from this g' = 0 to the origin is exactly the actual reliability index β'_A for the validation case. Simple trigonometric calculations show that this distance is indeed $(1+a)/(1+a^2)^{0.5}\times(1+a'^2)^{0.5}/(1+a')\times\beta_T$, as derived in Eq. (20).



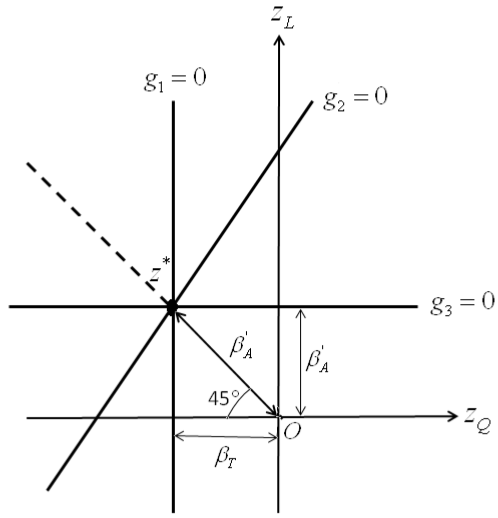


Fig. 5 (left) the scenario of a' = 1/a in QVM-z^*; (right) the extreme scenarios for QVM-z^*.

Note that β'_A does not depend on the mean values (μ_Q, μ_L) but only on the COVs (V_Q, V_L) through the ratios (a, a'), and $\beta'_A = \beta_T$ if the validation case has exactly the same ratio as the calibration case, i.e., $\xi'_L/\xi'_Q = \xi_L/\xi_Q$. More interestingly, $\beta'_A = \beta_T$ if a' = 1/a, i.e., $\xi'_Q/\xi'_L = \xi_L/\xi_Q$. This is somewhat less strict than DVM-z*, which requires a' = a to achieve the same target reliability index. This peculiar scenario is shown in the left plot in Fig. 5. In essence, if a' = 1/a, the limit state line for the validation case (g' = 0) will be a mirror image of that for the calibration case (g = 0) over the 1-to-(-1) line. As a result, $\beta'_A = \beta_T$.

In addition to QVM-z* achieving the ideal validation $\beta'_A = \beta_T$ under a less strict condition than DVM-z*, the more critical issue of unconservative design that happens in DVM-z* does not exist for QVM-z*. In fact, one can easily show (partly) by the Cauchy-Swartz Inequality that the ratio $(1+a)/(1+a)^{0.5}$ has a lower bound = 1 and an upper bound = $\sqrt{2}$. The consequence is that β'_A has a lower bound = $\beta_T/\sqrt{2}$ and an upper bound = $\beta_T\sqrt{2}$. The upper bound can happen if the limit state line for the calibration case is a perfectly vertical limit state line (i.e., $\xi_L = 0$), shown as $g_1 = 0$ in the right plot in Fig. 5, but the validation case has a limit state line inclines at 45^o (i.e., $\xi_Q = \xi_L$), shown as $g_2 = 0$ in the plot. In this special scenario, it is clear that $\beta'_A = \beta_T\sqrt{2}$. The lower bound can happen in the converse way – if the calibration case has a limit state line inclines at 45^o ($g_2 = 0$), but the validation case has a vertical limit state line ($g_1 = 0$). Surprisingly, if the calibration case has a vertical limit state line ($g_1 = 0$) but the validation case has a horizontal limit state line ($g_3 = 0$ in the right plot in Figure 5, i.e., $\xi_Q = 0$), perfect validation $\beta'_A = \beta_T$ will occur. For DVM-z*, the above scenario will give $\beta'_A = 0$!

Consider the same scenario where $V_L = 0.2$ and $V_Q = 0.3$ for the calibration case, and consider $V'_L = V_L$ and V'_Q ranges from 0.1 to 0.5 for the validation case, and the target is $\beta_T = 3.0$. The calibrated design location from Eq. (13) is (z^*_Q, γ^*_L) = (-2.161, 2.161). This means that for the validation case, one needs to comply with the algebraic design equation, $\lambda'_Q - \xi'_Q\times2.161 = \lambda'_L + \xi'_L\times2.161$ in the standard Gaussian space. Equivalently, one can state the design equation in the natural space of Q and L, $\gamma_Q\mu'_Q = \gamma_L\mu'_L$, where the partial factors (γ_Q, γ_L) now depend on the COVs for the validation case:

$$\gamma_Q = \exp\left(-0.5\xi'^2_Q - 2.161\xi'_Q\right) \qquad \gamma_L = \exp\left(-0.5\xi'^2_L + 2.161\xi'_L\right) \tag{21}$$

The resulting actual reliability index β'_A under various V'_Q is plotted as the thick solid line in the right plot in Fig. 3. It is clear that β'_A is fairly close to 3.0, ranging from 2.85 for $V'_Q = 0.1$ to 2.8 for $V'_Q = 0.5$. Also, β'_A is not always less than 3.0; in fact, $\beta'_A = 3.1$ reaches its maximum at $V'_Q = 0.2$.

4.3 More thorough comparisons

The thick solid lines in Fig. 3 only compares the robustness of standard DVM (or standard QVM), DVM-z*, and QVM-z* for the case with $V_Q = 0.3$. We say a reliability-based design equation is robust if it achieves the target reliability index for validation cases covering a wide range of design scenarios different from the calibration case. For other V_Q values, the results are shown in Fig. 3 as the grey lines. The number labels in the figure are the V_Q values. It is now clear that QVM-z^* performs the best in terms of maintaining β'_A to a value that is fairly close to the target value β_T = 3.0. The standard DVM or QVM performs the worst, indicating that applying constant partial factors calibrated for one case to another different case is not robust. The two strategies DVM-z^* and QVM-z^* with variable partial factors significantly outperform the standard DVM and QVM. The DVM-z^* approach always gives β'_A that is no greater than the target value β_T = 3.0. In some extreme scenarios, e.g., V_Q = 0.1 and V'_Q = 0.5, DVM-z^* performs quite poorly.

4.4 Most probable point versus uniform quantiles

The main difference between DVM-z^* and QVM-z^* is that the former uses the most probable point for Q and L on the limit state line, while the latter uses a point with equal exceedance/ non-exceedance probability for Q and L, referred to as the point with uniform quantiles in subsequent discussions. The former uses a design location z^* that is sensitive to the gradient of the limit state line, but the latter uses a design location z^* that is insensitive to the gradient. The gradient basically quantifies which variable (Q or L) dominates the overall uncertainty.

The problem with DVM-z^* is that the most probable scenario for the calibration case can hardly be the most probable design scenario for the validation case. In fact, the chance that these two scenarios coincide or are very similar is rather small. As a result, it is not sensible to design the validation case based on the most probable point for the calibration case. Unfortunately, under the framework of FORM, the actual reliability index β'_A for the validation case will be the same as the target value β_T only if the most probable point for the calibration case coincides with the most probable point for the validation case. The notion of "most probable point", although plausible and helpful in *reliability analysis* is counterproductive in *reliability-based code calibration*. It also explains why DVM-z^* is inferior to QVM-z^*.

On the contrary, QVM-z^* does not need to use a point that is "the most" in any sense. This is because the theory proposed in Ching and Phoon (2011) does not require optimizing any quantity. Moreover, the notion that a conservative design could be produced by reducing Q and increasing L to uniform quantiles makes sense for most cases and it is in line with how an engineer thinks intuitively.

Compared to using the FORM design point, using a design location with uniform quantiles has other advantages. In the previous example, only one case with V_Q = 0.3 is taken as the calibration case, and the validation case has V'_Q ranging from 0.1 to 0.5. If V'_Q for the validation cases covers such a wide range, it will make more sense to conduct the calibration based on more than one calibration case, e.g., use five calibration cases with V_Q = 0.1, 0.2, 0.3, 0.4, and 0.5. Under DVM-z^*, it is not straightforward to incorporate five calibration cases. One may find the FORM design points for the five cases and, say, take the centroid of those five design points. However, the distance between this centroid to the origin is not even β_T, so we cannot achieve the desired target value of β_T even approximately within the calibration domain. Ditlevsen and Madsen (1996) proposed a method involving solving an optimization problem for obtaining a unique design location for multiple calibration cases. A similar method proposed by Sørensen (2002) for obtaining the optimal partial factors will be discussed in a later section.

In contrast, it is very simple to incorporate five calibration cases under QVM-z^*: simply find the calibrated η values for the five cases and average them to get η_{ave}. The resulting design location is therefore $z^*_Q = \Phi^{-1}(\eta_{ave})$ and $z^*_L = -\Phi^{-1}(\eta_{ave})$. According to Eq. (12), the calibrated η values for the five calibration cases with V_Q = (0.1, 0.2, 0.3, 0.4, 0.5) are (0.0127, 0.0169, 0.0153, 0.0129, 0.0110), so the average η_{ave} is 0.0138. The resulting design location is therefore $z^*_Q = \Phi^{-1}(\eta_{ave}) = -2.203$ and $z^*_L = -\Phi^{-1}(\eta_{ave}) = 2.203$. The QVM-$z^*$ based on the average η_{ave} is therefore to assure the above point is on the limit state line for the validation case, i.e.,

$$\lambda'_Q - 2.203\xi'_Q - \lambda'_L - 2.203\xi'_L = 0 \tag{22}$$

As a result, the actual reliability index β'_A for the validation case is

$$\beta'_A = \frac{\lambda'_Q - \lambda'_L}{\sqrt{\xi'^2_Q + \xi'^2_L}} = \frac{2.203\left(\xi'_Q + \xi'_L\right)}{\sqrt{\xi'^2_Q + \xi'^2_L}} \tag{23}$$

The red line in the right plot in Fig. 3 shows the resulting actual reliability index β'_A under various V'_Q. It is clear that the resulting β'_A is fairly close to the target value 3.0.

4.5 Why choose the 1-to-(-1) line in QVM-z^*?

In QVM-z^*, the design location for the calibration case is the intersection of the 1-to-(-1) line and the adjusted limit state line g = 0. The choice of the 1-to-(-1) line seems somewhat arbitrary. In this section, the rationale for this choice will be demonstrated for the simple example. It is possible to use another line with a different angle. Let us consider using a line inclining with angle α to the horizontal z_Q axis, i.e., the 1-to-[-tan(α)] line. Due to the geometric symmetry, one only needs to consider the range of $-45^o \le \alpha \le 45^o$. As a result, the design location should be the intersection of the 1-to-[-tan(α)] line and the adjusted limit state line g = 0:

$$z^*_Q = \frac{-\beta_T\sqrt{\xi^2_Q + \xi^2_L}}{\xi_Q + \xi_L\tan\left(\alpha\right)} \qquad z^*_L = \frac{\beta_T\tan\left(\alpha\right)\sqrt{\xi^2_Q + \xi^2_L}}{\xi_Q + \xi_L\tan\left(\alpha\right)} \tag{24}$$

It can be shown that the β'_A for the validation case implied by the design location (z_Q^*, z_L^*) is

$$\beta'_A = \frac{1 + a\tan(\alpha)}{\sqrt{1+a^2}} \times \frac{\sqrt{1+a'^2}}{1 + a'\tan(\alpha)} \beta_T \tag{25}$$

One can easily show (partly) by the Cauchy-Swartz Inequality that the ratio $[1+a\times\tan(\alpha)]/(1+a)^{0.5}$ has a lower bound = $\tan(\alpha)$ and an upper bound = $1/\cos(\alpha)$ (note that $-45^o \le \alpha \le 45^o$). The consequence is that β'_A has a lower bound = $\sin(\alpha)$ and an upper bound = $1/\sin(\alpha)$. The difference between the two bounds is minimized by choosing $\alpha = 45^o$. As a result, the choice for the 1-to-(-1) line not only is intuitively plausible but is also optimal in the sense that the actual reliability index for a validation case has the tightest upper and lower bounds. The above results show that the choice of α is quite important. In the two realistic examples presented later on, it will be seen that as long as the principle "taking η quantiles for stabilizing variables and 1-η quantiles for destabilizing variables" is followed, QVM-z^* seems to be quite robust with regards to the choice of α.

CONCLUSION

In this paper, reliability-based code calibration methods based on quantiles (denoted by the quantile value method, QVM) and based on first-order reliability method (denoted by the design value method, DVM) are compared. The comparisons were made analytically and geometrically in the standard Gaussian space using a simple geotechnical design example to retain analytical tractability. The focus is on how robust the calibrated partial factors (or load and resistance factors) can be if these factors are to be applied to a design case that is not the same as the calibration case (specifically, COVs are different). The main conclusions are as follows:

1. Methods based on constant partial factors cannot be robust if COVs vary over a wide range in practice. That is to say, it is not robust to directly apply the partial factors calibrated for one case to another case. The resulting reliability index for the latter case may be very different from the target value. This conclusion is true for both QVM and DVM. The implication is that a design code with constant partial factors (or constant load and resistance factors) may not be robust for realistic geotechnical designs, because COVs of geo-materials indeed vary over a wide range. This issue may not be critical for a structural design code because the COVs of structural materials typically fall between 5% and 20%, but it is certainly critical for a geotechnical design code. In the companion paper, more realistic design examples will show that code calibration strategies based on constant partial factors are indeed not robust.

2. Applying the calibrated design location in the standard Gaussian space proves to be a more robust strategy than that based on constant partial factors. With this new strategy, the partial factors depend on the COVs of the geotechnical parameters. For DVM, the calibrated design location is the FORM design point in the standard Gaussian space, and the resulting method is denoted by DVM-z^*. For QVM, the

calibrated design location is a location in the standard Gaussian space that is never explored in the literature, and the resulting method is denoted by QVM-z^*.

3. Although DVM-z^* is more robust than the DVM based on constant partial factors, it suffers from a critical shortcoming of producing consistently unconservative design. This shortcoming is related to the principle of FORM – the FORM design point is the most probable point on the limit state line. The FORM design point calibrated for one case is suitable for another case *only if* this point is also the most probable point for the latter case. However, this is quite unlikely to happen. The QVM-z^* does not have this shortcoming. It will be made clear in the companion paper that this unconservative issue for DVM-z^* will be serious whenever the dominant random variable (one that dominates the overall uncertainty of the response) changes with design scenarios.

4. The QVM-z^* is based on the idea of uniform quantiles – reducing stabilizing random variables to their η quantiles and increasing destabilizing random variables to their 1-η quantiles. Robustness of this method is seriously compromised if stabilizing variables are increased while destabilizing variables are decreased, but this is human error which cannot be economically mitigated using any factors of safety.

5. It is easy to incorporate many calibration cases for QVM-z^*, although it is not as easy for DVM-z^*. Incorporating multiple calibration cases proves to provide extra robustness.

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