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Three-dimensional bending analysis of rectangular thick plates on elastic foundations using the B-spline Ritz method

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ABSTRACT

In the paper, three-dimensional elastic problems of the rectangular thick plates resting on elastic foundations were analyzed by using the B-spline Ritz method. The thick plates were subjected to a body force as the dead weight and a surface force as the fully distributed uniform load, and the elastic foundations were expressed by the Winkler model. Comparing numerical results of the B-spline Ritz method with the ones of analytical and finite element method, the present method was confirmed to have stable convergence and high accuracy. By analyzing some problems of the plate on foundation, the effects of thickness, modulus of subgrade reaction, transverse loading condition and boundary condition on the displacements and stresses of the plates resting on elastic foundations were clarified.

1. INTRODUCTION

A lot of engineering problems can be modeled as a thick plate on soil-foundation such as footing of buildings, pavement of roads, reinforced-concrete pavements of highways, airport runways, foundation of storage tanks and base of heavy machines, etc. Mechanical behaviors of a thick plate on elastic foundation are very important for low-cost and performance-based design.

The first-order shear deformation theory, namely Mindlin's theory (Mindlin 1951) is

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generally used to analyze a thick plate. In this theory, the transversely shear strain is assumed to be constant in the thickness direction. However, the transversely normal stress is neglected, and plays an important part to estimate mechanical strength of plate on foundation. Therefore, the analysis must be based on the three-dimensional (3-D) theory of elasticity with considering all stress components.

Rectangular thick plates on elastic foundations have been reported using the analytical and numerical method based on the Mindlin's theory. Kobayashi (1989) analyzed the rectangular plates on the Winkler foundations with two opposite simply supported and other two edges being arbitrary boundary conditions using the Lévy-type's analytical solution. Liew (1996) and Liu (2000) analyzed rectangular plates on the Winkler foundations with arbitrary boundary conditions using the differential quadrature method and differential quadrature element method, respectively. However, static analysis of rectangular plates resting on elastic foundation based on the 3-D theory of elasticity has not been reported in the literature.

In the paper, 3-D elastic problems of the rectangular thick plates resting on elastic foundations were analyzed by using the B-spline Ritz method. The B-spline Ritz method has been proposed by Nagino (2008). This method is formulated by the Ritz procedure with triplicate series of normalized B-spline functions (Boor 1972) as displacement components. The thick plates were subjected to a body force as the dead weight and a surface force as the fully distributed uniform load, and the elastic foundations were expressed by the Winkler model. Comparing numerical results of the B-spline Ritz method with the ones of an analytical solution and the finite element method, the present method was confirmed to have stable convergence and high accuracy. By analyzing some problems of the plate on foundation, the effects of thickness, modulus of subgrade reaction, transverse loading condition and boundary condition on the displacements and stresses of the square thick plates resting on elastic foundations were clarified.

2. MATHEMATICAL FORMULATION

2.1. Analytical model

Consider a rectangular plate with length a, width b and uniform thickness h, which is resting on an elastic foundation as shown in Figure 1. The plate is defined with respect to a right-handed orthogonal coordinate system (x, y, z). The displacement components at any point are defined by the in-plane components u, v and the transverse component w in the x, y and z direction, respectively. The horizontal displacement components in the foundation are assumed to be negligible, and surface of foundation is smooth. In this paper, the Winkler model is used. This model is expressed as discrete elastic spring having the modulus of subgrade reaction k_1.

Navier's equations considering transverse body force Z for elastic rectangular palte are given by

$$\nabla^2 u + \frac{1}{1-2\nu}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \, ,$$

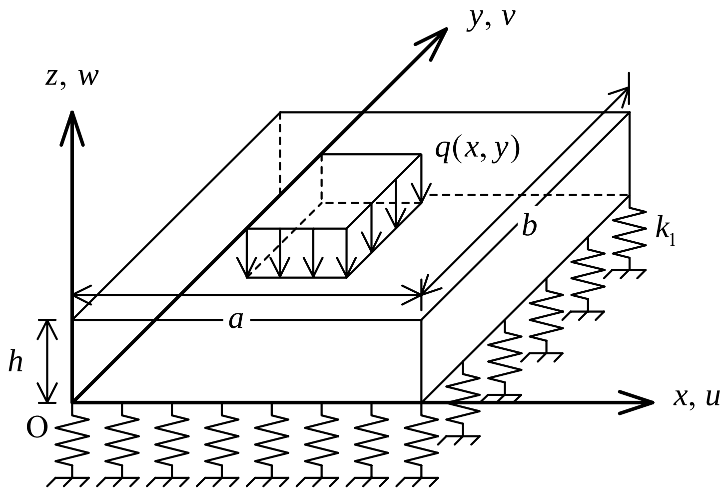


Figure 1. Coordinate system and geometry of rectangular plate on elastic foundation

$$\nabla^2 v + \frac{1}{1-2\nu}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0,$$
$$\nabla^2 w + \frac{1}{1-2\nu}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\frac{Z}{G}=0, \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}, \quad G=\frac{E}{2(1+\nu)} \tag{2}$$

in which ∇^2 is Laplacian, ν is Poisson's ration, E is Young's modulus and G is shear modulus.

In the 3-D theory of elasticity, strain and stress components are defined as

$$\varepsilon_x=\frac{\partial u}{\partial x},\ \varepsilon_y=\frac{\partial v}{\partial y},\ \varepsilon_z=\frac{\partial w}{\partial z},\ \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\ \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\ \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}, \tag{3}$$

$$\sigma_x=\lambda e+2\mu\varepsilon_x,\ \sigma_y=\lambda e+2\mu\varepsilon_y,\ \sigma_z=\lambda e+2\mu\varepsilon_z,$$
$$\tau_{xy}=\mu\gamma_{xy},\ \tau_{yz}=\mu\gamma_{yz},\ \tau_{zx}=\mu\gamma_{zx}, \tag{4}$$

where

$$\lambda=\frac{2\nu}{1-2\nu}\mu,\quad \mu=G,\quad e=\varepsilon_x+\varepsilon_y+\varepsilon_z, \tag{5}$$

in which λ and μ are Lamé's constants, and e is volumetric strain.

The boundary conditions at the four edges ($x = 0$, a and $y = 0$, b) of a rectangular plate would be satisfied as follows:

Simply supported:

$$v = w = 0, \ \ \sigma_x = 0 \ \text{ at } \ (x = 0, a),$$
$$u = w = 0, \ \ \sigma_y = 0 \ \text{ at } \ (y = 0, b). \tag{6}$$

Clamped edge:

$$u = v = w = 0 \ \text{ at } \ (x = 0, a),$$
$$u = v = w = 0 \ \text{ at } \ (y = 0, b). \tag{7}$$

Free edge:

$$\sigma_x = \tau_{xy} = \tau_{xz} = 0 \ \text{ at } \ (x = 0, a),$$
$$\sigma_y = \tau_{yx} = \tau_{yz} = 0 \ \text{ at } \ (y = 0, b). \tag{8}$$

The boundary conditions for top and bottom surfaces of the plate can be expressed by

$$\sigma_z = -q\,, \ \tau_{yz} = \tau_{zx} = 0 \quad \text{at} \quad (z = h)\,,$$
$$\sigma_z = k_1 w\,, \ \tau_{yz} = \tau_{zx} = 0 \quad \text{at} \quad (z = 0)\,. \tag{9}$$

2.2. Formulation of governing equation based on the B-spline Ritz method

The strain energy $\bar{U}$ of a rectangular plate on elastic foundation can be expressed in integral form as

$$\bar{U} = \frac{1}{2}\int_0^a \int_0^b \int_0^h (\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$
$$+\frac{k_1}{2}\int_0^a \int_0^b (w|_{z=0})^2\,\mathrm{d}y\,\mathrm{d}x\,. \tag{10}$$

The potential of external force $\bar{V}$ can be written as

$$\bar{V} = \int_0^a \int_0^b q\,w|_{z=h}\,\mathrm{d}y\,\mathrm{d}x + \int_0^a \int_0^b \int_0^h Zw\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,. \tag{11}$$

Here, for simplicity and convenience in mathematical formulation, the following non-dimensional coordinate system (ξ, η, ζ) are introduced as

$$\xi = \frac{x}{a}\,, \ \eta = \frac{y}{b}\,, \ \zeta = \frac{z}{h}\,. \tag{12}$$

The displacement components can be expressed by non-dimensional displacement function U, V and W in ξ, η and ζ directions, respectively, as

$$u(x,y,z) = aU(\xi,\eta,\zeta)\,,\ v(x,y,z) = aV(\xi,\eta,\zeta)\,,\ w(x,y,z) = aW(\xi,\eta,\zeta)\,. \tag{13}$$

The assumed spatial displacement field is based on a separable assumption for displacement functions. The functions are expressed as the summation of a triplicate series of B-spline functions as follows:

$$\begin{aligned} U(\xi,\eta,\zeta) &= \sum_{m=1}^{i_\xi}\sum_{n=1}^{i_\eta}\sum_{r=1}^{i_\zeta} A_{mnr} N_{m,k_\xi}(\xi) N_{n,k_\eta}(\eta) N_{r,k_\zeta}(\zeta)\,, \\ V(\xi,\eta,\zeta) &= \sum_{m=1}^{i_\xi}\sum_{n=1}^{i_\eta}\sum_{r=1}^{i_\zeta} B_{mnr} N_{m,k_\xi}(\xi) N_{n,k_\eta}(\eta) N_{r,k_\zeta}(\zeta)\,, \\ W(\xi,\eta,\zeta) &= \sum_{m=1}^{i_\xi}\sum_{n=1}^{i_\eta}\sum_{r=1}^{i_\zeta} C_{mnr} N_{m,k_\xi}(\xi) N_{n,k_\eta}(\eta) N_{r,k_\zeta}(\zeta)\,, \end{aligned} \tag{14}$$

in which $N_{m,k_\xi}(\xi)$, $N_{n,k_\eta}(\eta)$ and $N_{r,k_\zeta}(\zeta)$ are one-dimensional (1-D) normalized B-spline functions with the degree of spline function $(k_\xi - 1)$, $(k_\eta - 1)$ and $(k_\zeta - 1)$. A_{mnr}, B_{mnr} and C_{mnr} are unknown spline coefficients. The appearing in Eq. (14) are defined as: $i_{k_\xi} = m_\xi + k_\xi - 2$, $i_{k_\eta} = m_\eta + k_\eta - 2$ and $i_{k_\zeta} = m_\zeta + k_\zeta - 2$, where m_ξ, m_η, m_ζ and k_ξ, k_η, k_ζ are the number of knots and the order of spline function in the ξ, η, ζ directions, respectively.

Substituting Eqs. (12), (13) and (14) into Eqs. (10) and (11), the strain energy $\bar{U}$ and potential of external force $\bar{V}$ can be written in a non-dimensional coordinate systems as

$$\begin{aligned} \bar{U} &= \frac{abhE}{2}\left\{\int_0^1\int_0^1\int_0^1 (\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})\,\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}\xi \right. \\ &\quad \left. + \Theta\left(\frac{a}{h}\right)\int_0^1\int_0^1 (W|_{\zeta=0})^2\,\mathrm{d}\eta\,\mathrm{d}\xi \right\} \\ &= \frac{abhE}{2}\{\Delta\}^{\mathrm{T}}([\mathrm{K_P}] + [\mathrm{K_F}])\{\Delta\}\,, \end{aligned} \tag{15}$$

$$\begin{aligned} \bar{V} &= a^2bq\int_0^1\int_0^1 W|_{\zeta=1}\,\mathrm{d}\eta\,\mathrm{d}\xi + a^2bhZ\int_0^1\int_0^1\int_0^1 W\,\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &= a^2b[q_0\{\delta_C\}^{\mathrm{T}}\{\mathrm{f_q}\} + q_Z\{\delta_C\}^{\mathrm{T}}\{\mathrm{f_Z}\}]\,, \end{aligned} \tag{16}$$

$$\Theta = \frac{k_1 a}{E}\,, \tag{17}$$

where $q_Z = Zh$ is load strength of a body force Z, q_0 is load strength of a surface force q, Θ is non-dimensional foundation parameter, $[\mathrm{K_P}]$ is the stiffness matrix of the plate, and $[\mathrm{K_F}]$ is the stiffness matrix of the foundation. Here, $\{\mathrm{f_q}\}$ is force vector for a surface force q_0, and $\{\mathrm{f_Z}\}$ is force vector for body force Z. $\{\Delta\}$ is unknown coefficient vector in the following:

$$\{\Delta\} = \left\{ \{\delta_A\} \quad \{\delta_B\} \quad \{\delta_C\} \right\}^{\mathrm{T}}, \tag{18}$$

in which the column vectors $\{\delta_A\}$, $\{\delta_B\}$ and $\{\delta_C\}$ as

$$\begin{aligned}
\{\delta_A\} &= \{A_{111} \quad A_{112} \quad \cdots \quad A_{11i_\zeta} \quad A_{121} \quad \cdots \quad A_{12i_\zeta} \quad \cdots \quad A_{1i_\eta i_\zeta} \quad \cdots \quad A_{i_\xi i_\eta i_\zeta}\}^{\mathrm{T}}, \\
\{\delta_B\} &= \{B_{111} \quad B_{112} \quad \cdots \quad B_{11i_\zeta} \quad B_{121} \quad \cdots \quad B_{12i_\zeta} \quad \cdots \quad B_{1i_\eta i_\zeta} \quad \cdots \quad B_{i_\xi i_\eta i_\zeta}\}^{\mathrm{T}}, \\
\{\delta_C\} &= \{C_{111} \quad C_{112} \quad \cdots \quad C_{11i_\zeta} \quad C_{121} \quad \cdots \quad C_{12i_\zeta} \quad \cdots \quad C_{1i_\eta i_\zeta} \quad \cdots \quad C_{i_\xi i_\eta i_\zeta}\}^{\mathrm{T}}.
\end{aligned} \tag{19}$$

The total potential energy Π of the rectangular plate on elastic foundation can be expressed as

$$\Pi = \bar{U} - \bar{V}. \tag{20}$$

In Eq. (20), minimizing the total potential energy Π with respect to the unknown spline coefficient vector $\{\Delta\}$ i.e.:

$$\frac{\partial \Pi}{\partial \{\Delta\}} = 0, \tag{21}$$

which lead to the following the governing equation in matrix form:

$$\begin{bmatrix} [\mathrm{K_{UU}}] & [\mathrm{K_{UV}}] & [\mathrm{K_{UW}}] \\ [\mathrm{K_{VU}}] & [\mathrm{K_{VV}}] & [\mathrm{K_{VW}}] \\ [\mathrm{K_{WU}}] & [\mathrm{K_{WV}}] & [\mathrm{K_{WW}}]+[\mathrm{K_{WW}^{Wink}}] \end{bmatrix} \begin{Bmatrix} \{\delta_A\} \\ \{\delta_B\} \\ \{\delta_C\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{\mathrm{f_q}\} \end{Bmatrix} + \begin{Bmatrix} \{0\} \\ \{0\} \\ \{\mathrm{f_Z}\} \end{Bmatrix}, \tag{22}$$

in which $[\mathrm{K_{IJ}}]$ $(\mathrm{I, J = U, V, W})$ is the sub-stiffness matrices of the plate and $[\mathrm{K_{WW}^{Wink}}]$ is the sub-stiffness matrices of the Winkler type's elastic foundation. The size of matrix in Eq. (22) is $3(m_\xi + k_\xi - 2)(m_\eta + k_\eta - 2)(m_\zeta + k_\zeta - 2)$.

3. NUMERICAL RESULTS

The displacements and stresses of rectangular thick plates resting on elastic foundations with arbitrary boundary conditions were analyzed. The plate is made of concrete, mechanical properties of the plate were as follows; Young's modulus $E = 23.1$

Table 1. Convergence and comparison of displacements $wE / q_Z a$ and stresses σ_x / q_Z for SS-SS square thick plates on elastic foundation under body force Z.

h/a	$(k_\xi - 1)\times(k_\eta - 1)\times(k_\zeta - 1)$	$m_\xi \times m_\eta \times m_\zeta$	$\frac{wE}{q_Z a}$		$\frac{\sigma_x}{q_Z} = \frac{\sigma_y}{q_Z}$	
			$(\zeta = 0)$	$(\zeta = 1)$	$(\zeta = 0)$	$(\zeta = 1)$
0.2	$3\times3\times3$	$11\times11\times3$	-6.497	-6.504	6.440	-6.449
		$21\times21\times5$	-6.497	-6.504	6.429	-6.437
	$4\times4\times4$	$11\times11\times3$	-6.497	-6.504	6.430	-6.438
		$21\times21\times5$	-6.497	-6.504	6.430	-6.438
	Analytical solution	–	-6.497	-6.504	6.430	-6.438
	3-D FEM (C3D8)	$51\times51\times11$	-6.506	-6.513	5.912	-5.920
		$101\times101\times21$	-6.500	-6.506	6.166	-6.175
	3-D FEM (C3D20)	$21\times21\times5$	-6.497	-6.503	6.443	-6.452
		$51\times51\times11$	-6.490	-6.434	6.432	-6.441
0.5	$3\times3\times3$	$13\times13\times7$	-0.7517	-0.7535	1.173	-1.175
		$17\times17\times9$	-0.7517	-0.7535	1.173	-1.175
		$21\times21\times11$	-0.7517	-0.7535	1.173	-1.175
		$25\times25\times13$	-0.7517	-0.7535	1.173	-1.175
	$4\times4\times4$	$13\times13\times7$	-0.7517	-0.7535	1.173	-1.175
		$17\times17\times9$	-0.7517	-0.7535	1.173	-1.175
		$21\times21\times11$	-0.7517	-0.7535	1.173	-1.175
		$25\times25\times13$	-0.7517	-0.7535	1.173	-1.175
	Analytical solution	–	-0.7517	-0.7535	1.173	-1.175
	3-D FEM (C3D8)	$41\times41\times21$	-0.7514	-0.7532	1.109	-1.111
		$69\times69\times35$	-0.7516	-0.7534	1.135	-1.137
	3-D FEM (C3D20)	$29\times29\times15$	-0.7517	-0.7535	1.174	-1.176
		$41\times41\times21$	-0.7517	-0.7535	1.173	-1.176

GPa and Poisson's ration $\nu = 0.2$. The plates were subjected to a body force Z as the dead weight and a surface force q as the fully distributed uniform load. For the definition of the boundary conditions of the plate, for example, the symbols SF-CS, identifies a plate with edges ($\xi = 0, 1$) and ($\eta = 0, 1$) having simply supported edge (S), free edge (F), clamped edge (C) and simply supported edge (S), respectively. The placement of the knots was set to the Chebyshev-Gauss-Lobatto points, namely the shifted Chebyshev points (Nagino 2008) in the following analysis. The convergence and accuracy of the present method were investigated. Furthermore, the effects of thickness-to-length ratio h / a, non-dimensional foundation parameter Θ, transverse loading condition and boundary condition on the displacements and stresses of square thick plates ($b / a = 1$) on elastic foundations were also investigated.

All computations are performed in double precision on a personal computer, and all of the displacements and stresses are organized to four significant digits.

Table 2. Convergence of and comparison displacements $wE / q_Z a$ and stresses σ_x / q_Z for CC-CC square thick plates on elastic foundation under body force Z.

h/a	$(k_\xi - 1)\times(k_\eta - 1)\times(k_\zeta - 1)$	$m_\xi \times m_\eta \times m_\zeta$	$\frac{wE}{q_Z a}$		$\frac{\sigma_x}{q_Z} = \frac{\sigma_y}{q_Z}$	
			$(\zeta = 0)$	$(\zeta = 1)$	$(\zeta = 0)$	$(\zeta = 1)$
0.2	$3\times3\times3$	$11\times11\times3$	−2.820	−2.823	3.305	−3.311
		$21\times21\times5$	−2.825	−2.827	3.292	−3.298
	$4\times4\times4$	$11\times11\times3$	−2.825	−2.828	3.293	−3.299
		$21\times21\times5$	−2.826	−2.828	3.294	−3.299
	3-D FEM (C3D8)	$51\times51\times11$	−2.818	−2.821	3.006	−3.012
		$101\times101\times21$	−2.823	−2.826	3.148	−3.153
	3-D FEM (C3D20)	$21\times21\times5$	−2.816	−2.819	3.309	−3.314
		$51\times51\times11$	−2.824	−2.826	3.296	−3.302
0.5	$3\times3\times3$	$13\times13\times7$	−0.5158	−0.5170	0.6496	−0.6514
		$17\times17\times9$	−0.5158	−0.5170	0.6490	−0.6509
		$21\times21\times11$	−0.5159	−0.5170	0.6489	−0.6508
		$25\times25\times13$	−0.5159	−0.5170	0.6489	−0.6508
	$4\times4\times4$	$13\times13\times7$	−0.5159	−0.5170	0.6485	−0.6503
		$17\times17\times9$	−0.5159	−0.5170	0.6488	−0.6506
		$21\times21\times11$	−0.5159	−0.5170	0.6489	−0.6507
		$25\times25\times13$	−0.5159	−0.5171	0.6489	−0.6507
	3-D FEM (C3D8)	$41\times41\times21$	−0.5148	−0.5160	0.6045	−0.6062
		$69\times69\times35$	−0.5154	−0.5166	0.6226	−0.6243
	3-D FEM (C3D20)	$29\times29\times15$	−0.5155	−0.5166	0.6504	−0.6522
		$41\times41\times21$	−0.5157	−0.5168	0.6496	−0.6514

3.1. Convergence and comparison studies

Tables 1 and 2 show the effects of the degree of spline functions and the number of knots on the convergence of the displacements $wE / q_Z a$ and stresses σ_x / q_Z at $(\zeta = 0)$ and $(\zeta = 1)$ for square thick plate resting on elastic foundation under transverse body force Z having SS-SS and CC-CC. The thickness-to-length ratio h / a is set as 0.2 (moderately thick plate) and 0.5 (thick plate). The non-dimensional foundation parameter Θ is set to be 10^{-2}. The results are compared with the analytical solutions and the 3-D finite element solutions by Abaqus 6.12. Here, C3D8 and C3D20 mean first-order solid element and second-order solid element, respectively.

Tables 1 and 2 show that stable convergence can be obtained by increasing the number of knots. It is found that displacements and stresses rapidly converge by using the degree of spline functions $(k_\xi - 1)\times(k_\eta - 1)\times(k_\zeta - 1) = 4\times4\times4$. The results in Tables 1 and 2 show excellent agreement in all cases.

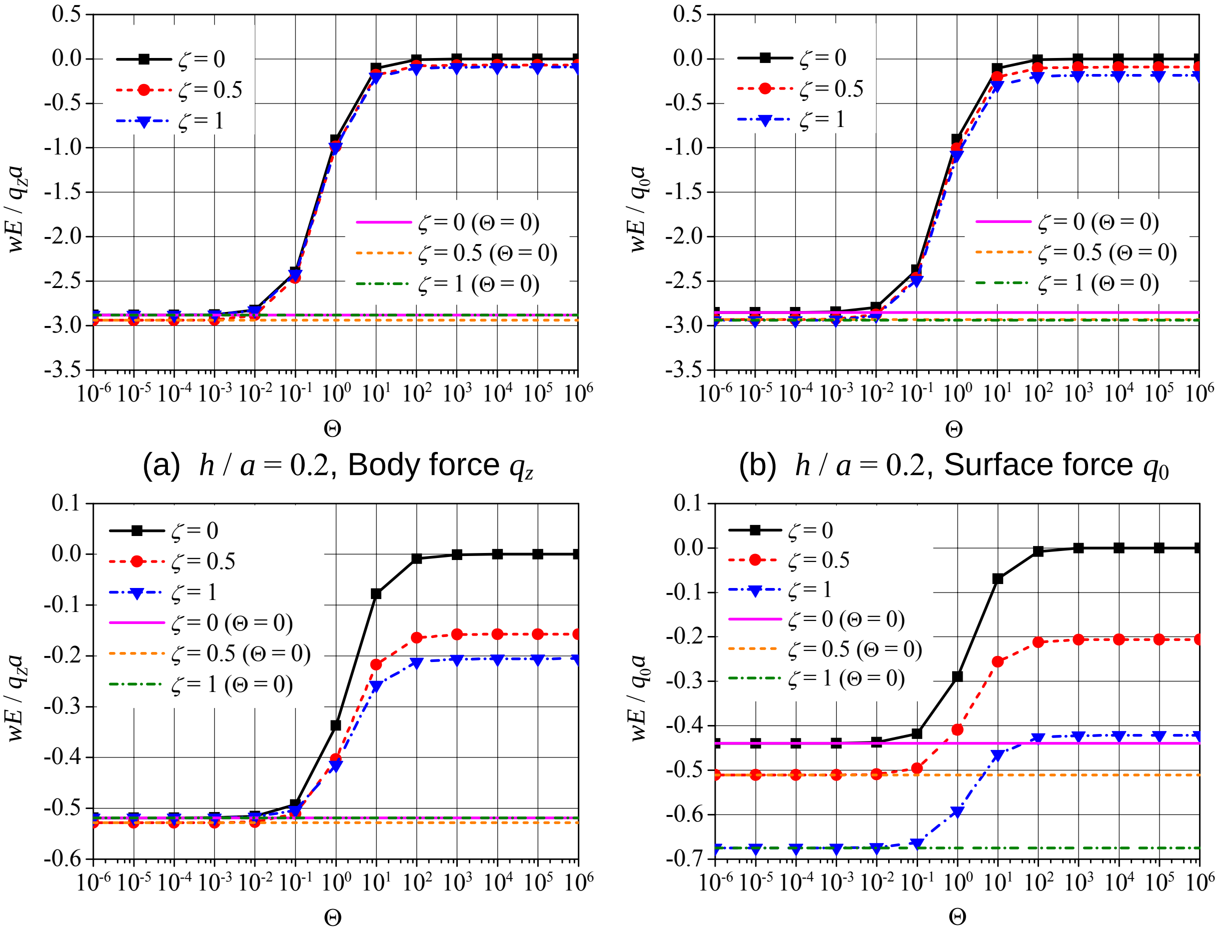
Tables 3 and 4 show the effects of the number of knots on the convergence of the displacements $wE / q_0 a$ and stresses σ_x / q_0 at $(\zeta = 0)$ and $(\zeta = 1)$ for square thick plate resting on elastic foundation under surface force q as the fully distributed uniform load

Table 3. Convergence and comparison of displacements $wE / q_0 a$ and stresses σ_x / q_0 for SS-SS square thick plates on elastic foundation under surface force.

h/a	Solution methods	$m_\xi \times m_\eta \times m_\zeta$	$\frac{wE}{q_0 a}$ ($\zeta=0$)	$\frac{wE}{q_0 a}$ ($\zeta=1$)	$\frac{\sigma_x}{q_0}=\frac{\sigma_y}{q_0}$ ($\zeta=0$)	$\frac{\sigma_x}{q_0}=\frac{\sigma_y}{q_0}$ ($\zeta=1$)
0.2	Present	11×11×3	– 6.386	– 6.488	6.436	– 6.544
		21×21×5	– 6.386	– 6.488	6.436	– 6.544
		31×31×7	– 6.386	– 6.488	6.436	– 6.545
	Analytical solution	–	– 6.386	– 6.488	6.436	– 6.545
	3-D FEM (C3D8)	51×51×11	– 6.395	– 6.498	5.911	– 6.020
		101×101×21	– 6.388	– 6.491	6.169	– 6.277
	3-D FEM (C3D20)	21×21×5	– 6.385	– 6.488	6.449	– 6.558
		51×51×11	– 6.386	– 6.488	6.438	– 6.547
0.5	Present	13×13×7	– 0.6398	– 0.8923	1.154	– 1.296
		17×17×9	– 0.6398	– 0.8923	1.154	– 1.296
		21×21×11	– 0.6398	– 0.8923	1.154	– 1.296
	Analytical solution	–	– 0.6398	– 0.8923	1.154	– 1.296
	3-D FEM (C3D8)	41×41×21	– 0.6394	– 0.8919	1.087	– 1.229
		69×69×35	– 0.6397	– 0.8922	1.114	– 1.256
	3-D FEM (C3D20)	29×29×15	– 0.6398	– 0.8923	1.156	– 1.296
		41×41×21	– 0.6398	– 0.8923	1.155	– 1.296

Table 4. Convergence and comparison of displacements $wE / q_0 a$ and stresses σ_x / q_0 for CC-CC square thick plates on elastic foundation under sueface force.

h/a	Solution methods	$m_\xi \times m_\eta \times m_\zeta$	$\frac{wE}{q_0 a}$ ($\zeta=0$)	$\frac{wE}{q_0 a}$ ($\zeta=1$)	$\frac{\sigma_x}{q_0}=\frac{\sigma_y}{q_0}$ ($\zeta=0$)	$\frac{\sigma_x}{q_0}=\frac{\sigma_y}{q_0}$ ($\zeta=1$)
0.2	Present	11×11×3	– 2.792	– 2.885	3.297	– 3.537
		21×21×5	– 2.794	– 2.887	3.298	– 3.540
		31×31×7	– 2.795	– 2.888	3.299	– 3.541
	3-D FEM (C3D8)	51×51×11	– 2.786	– 2.879	2.999	– 3.240
		101×101×21	– 2.792	– 2.885	3.147	– 3.388
	3-D FEM (C3D20)	21×21×5	– 2.784	– 2.877	3.313	– 3.554
		51×51×11	– 2.792	– 2.886	3.301	– 3.543
0.5	Present	13×13×7	– 0.4373	– 0.6738	0.6356	– 0.8993
		17×17×9	– 0.4373	– 0.6737	0.6361	– 0.8997
		21×21×11	– 0.4374	– 0.6737	0.6362	– 0.8998
		25×25×13	– 0.4374	– 0.6738	0.6362	– 0.8998
	3-D FEM (C3D8)	41×41×21	– 0.4357	– 0.6723	0.5864	– 0.8480
		69×69×35	– 0.4367	– 0.6731	0.6066	– 0.8692
	3-D FEM (C3D20)	29×29×15	– 0.4366	– 0.6731	0.6378	– 0.9004
		41×41×21	– 0.4370	– 0.6734	0.6370	– 0.9000



(a) $h / a = 0.2$, Body force q_z

(b) $h / a = 0.2$, Surface force q_0

(c) $h / a = 0.5$, Body force q_z

(d) $h / a = 0.5$, Surface force q_0

Figure 2. The effects of thickness-to-length ratio, non-dimensional foundation parameter and loading condition on the transverse displacements of CC-CC square thick plates on elastic foundations.

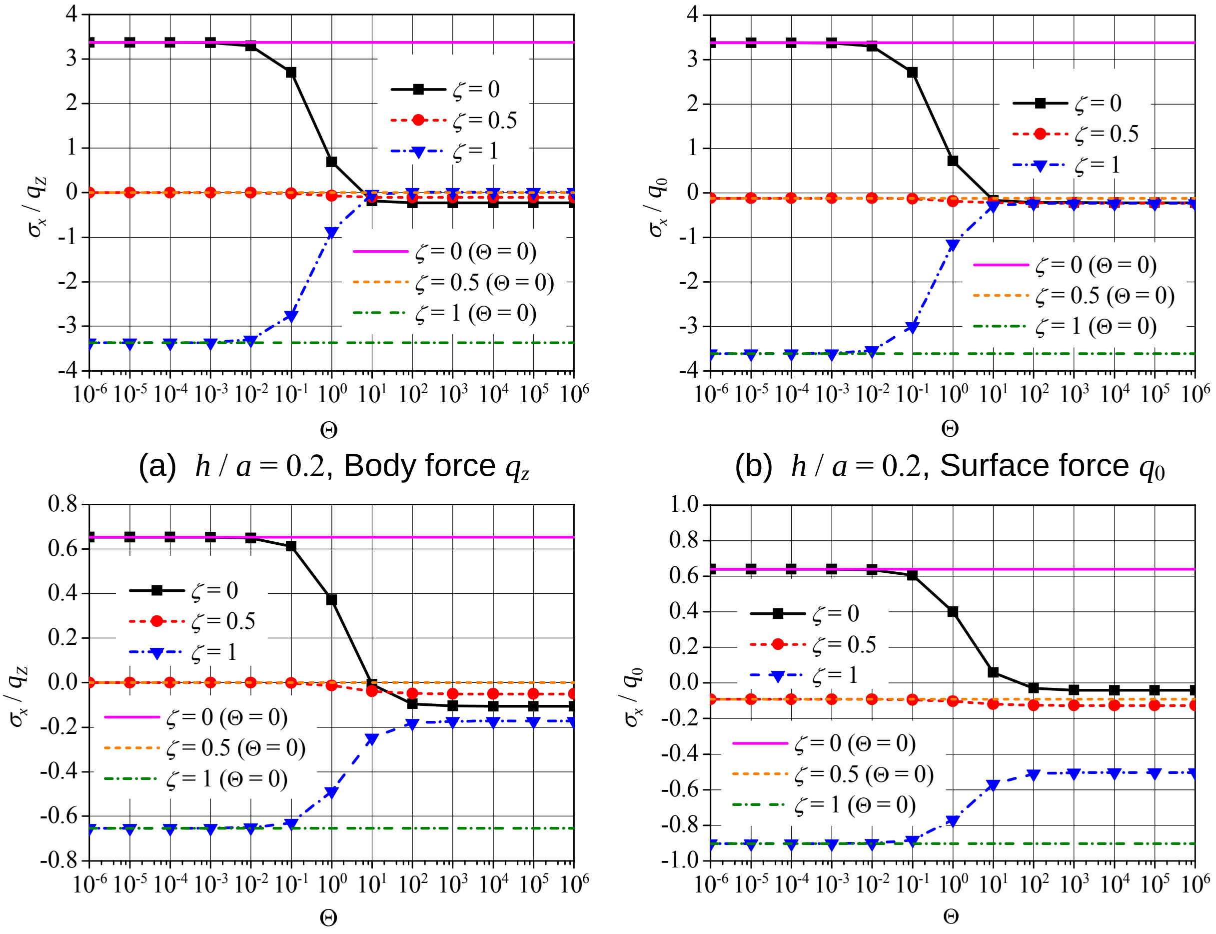
q_0 having SS-SS and CC-CC. The numerical calculation is same condition as Tables 1 and 2.

Tables 3 and 4 also show that stable convergence can be obtained, and high accurate results are also obtained in all cases.

3.2. Results and discussions

Figure 2 shows the effects of thickness-to-length ratio h / a, non-dimensional foundation parameter Θ and loading condition on the transverse displacement $wE / q_\Gamma a$ ($\Gamma = Z, 0$) of CC-CC square thick plates on elastic foundations. The non-dimensional foundation parameter Θ varies from 10^{-6} to 10^{6}. Note that $\Theta = 0$ means no foundation condition. The thickness-to-length ratio h / a is set as 0.2 and 0.5.

For $\Theta > 10^2$, the transverse displacement at bottom surface ($\zeta = 0$) is zero, namely the elastic foundation is nearly rigid compared to the plate. For range of $\Theta < 10^{-2}$, the



(a) $h / a = 0.2$, Body force q_z (b) $h / a = 0.2$, Surface force q_0

(c) $h / a = 0.5$, Body force q_z (d) $h / a = 0.5$, Surface force q_0

Figure 3. The effects of thickness-to-length ratio, non-dimensional foundation parameter and loading condition on the in-plane normal stress of square thick plates on elastic foundations.

transverse displacement is constant, and this is the same results as the no foundation ($\Theta = 0$). In the case of moderately thick plate, the transverse displacement along the thickness direction for each Θ is about the same regardless position along the thickness direction. On the other hand, in the case of thick plate, the transverse displacement is different depend on the position along the thickness direction.

The effects of thickness-to-length ratio h / a, non-dimensional foundation parameter Θ and loading condition on in-plane normal stress σ_x / q_Γ ($\Gamma = Z, 0$) of CC-CC square thick plates on elastic foundations are shown in Figure 3. The numerical calculation is same condition as the Figure 2.

With increasing Θ, the in-plane normal stress at top and bottom surfaces ($\zeta = 0$ and $\zeta = 1$) are decrease regardless of thickness-to-length ratio and transverse loading condition. For range of $\Theta < 10^{-2}$ and $\Theta > 10^2$, the in-plane normal stress is constant.

Therefore, the elastic foundation of range of $\Theta > 10^2$ is a rigid foundation, and range

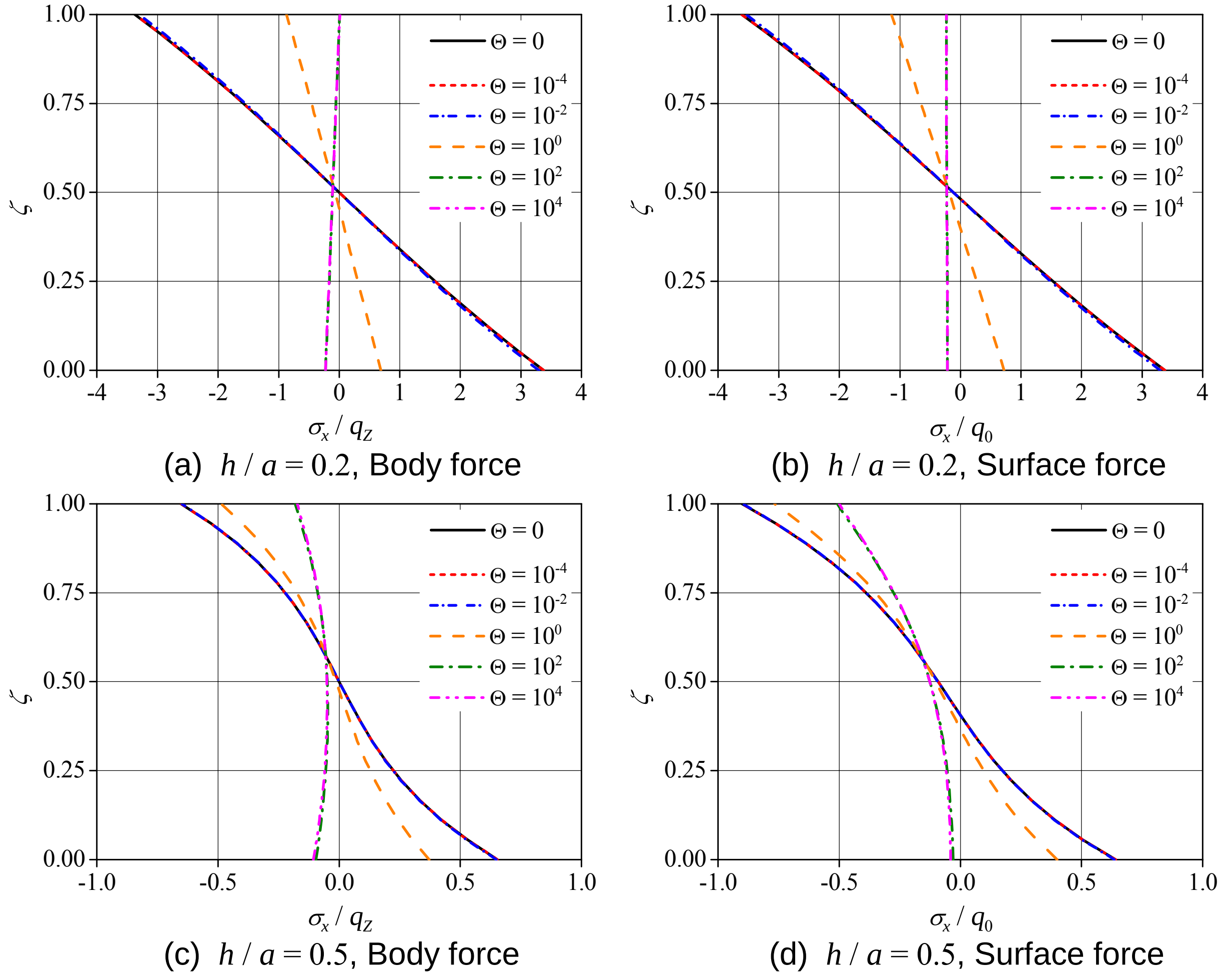
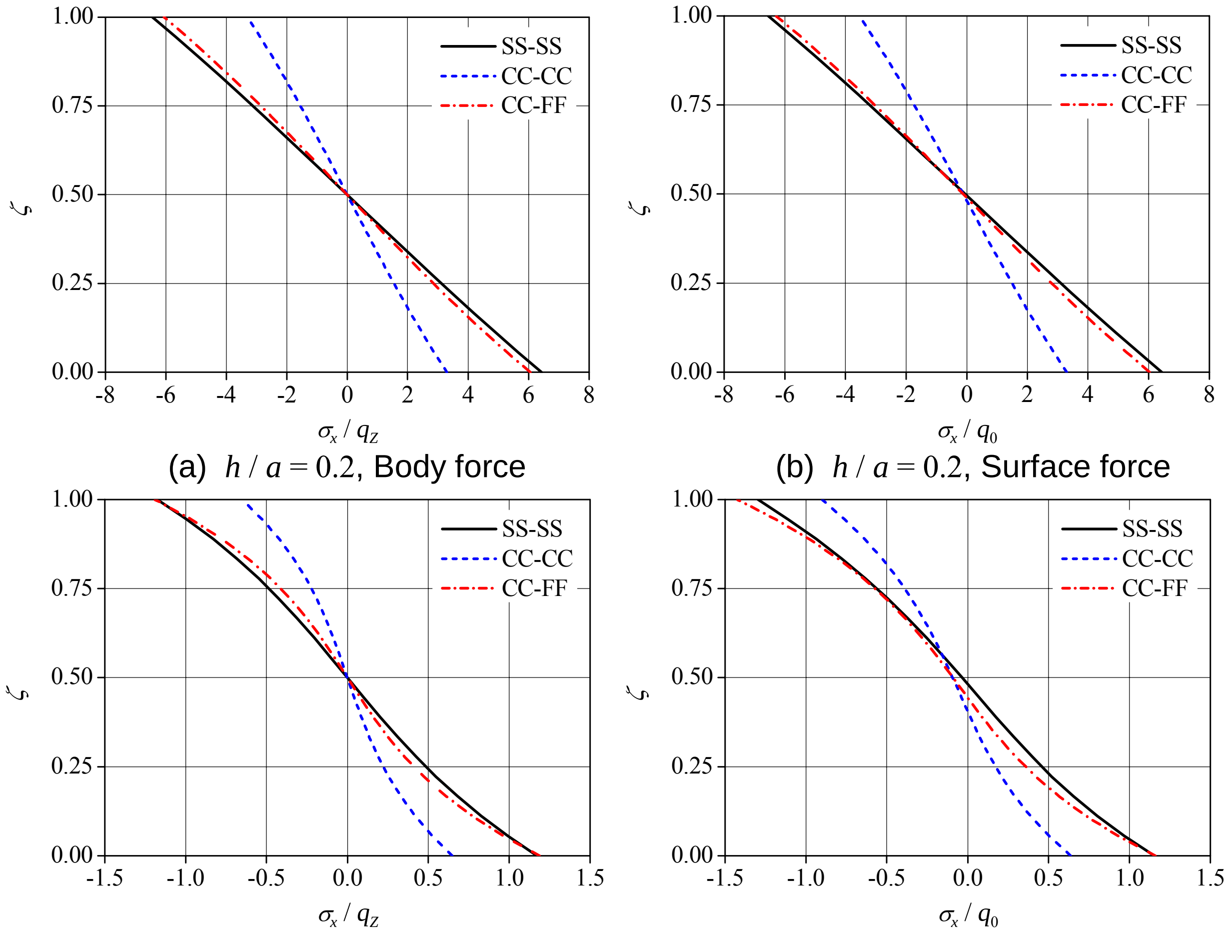


Figure 4. The effects of thickness-to-length ratio, non-dimensional foundation parameter and loading condition on distribution of the in-plane normal stress of square thick plates on elastic foundations.

of $\Theta < 10^{-2}$ is same as the results of plate without foundation. In the results, the effect of the foundation can be neglected in the analysis for $\Theta < 10^{-2}$.

Figure 4 shows the effects of thickness-to-length ratio, non-dimensional foundation parameter and loading condition on distribution of the in-plane normal stress σ_x / q_Γ ($\Gamma = Z, 0$) in the thickness direction of square thick plates on elastic foundations. The non-dimensional foundation parameter Θ varies from 10^{-4} to 10^{4}. Note that $\Theta = 0$ means no foundation condition. The thickness-to-length ratio h / a is set as 0.2 and 0.5.

With increasing Θ, the normal stresses are reduced regardless of thickness and transverse loading condition. The distribution of normal stress in the thickness direction for moderately thick plate is linearly. On the other hand, in the case of thick plate, distribution of normal stress is non-linearly, and the neutral surface of normal stress has been moved down. This cause is considered to be the effects of the transverse shear deformation and the surface force acting locally.



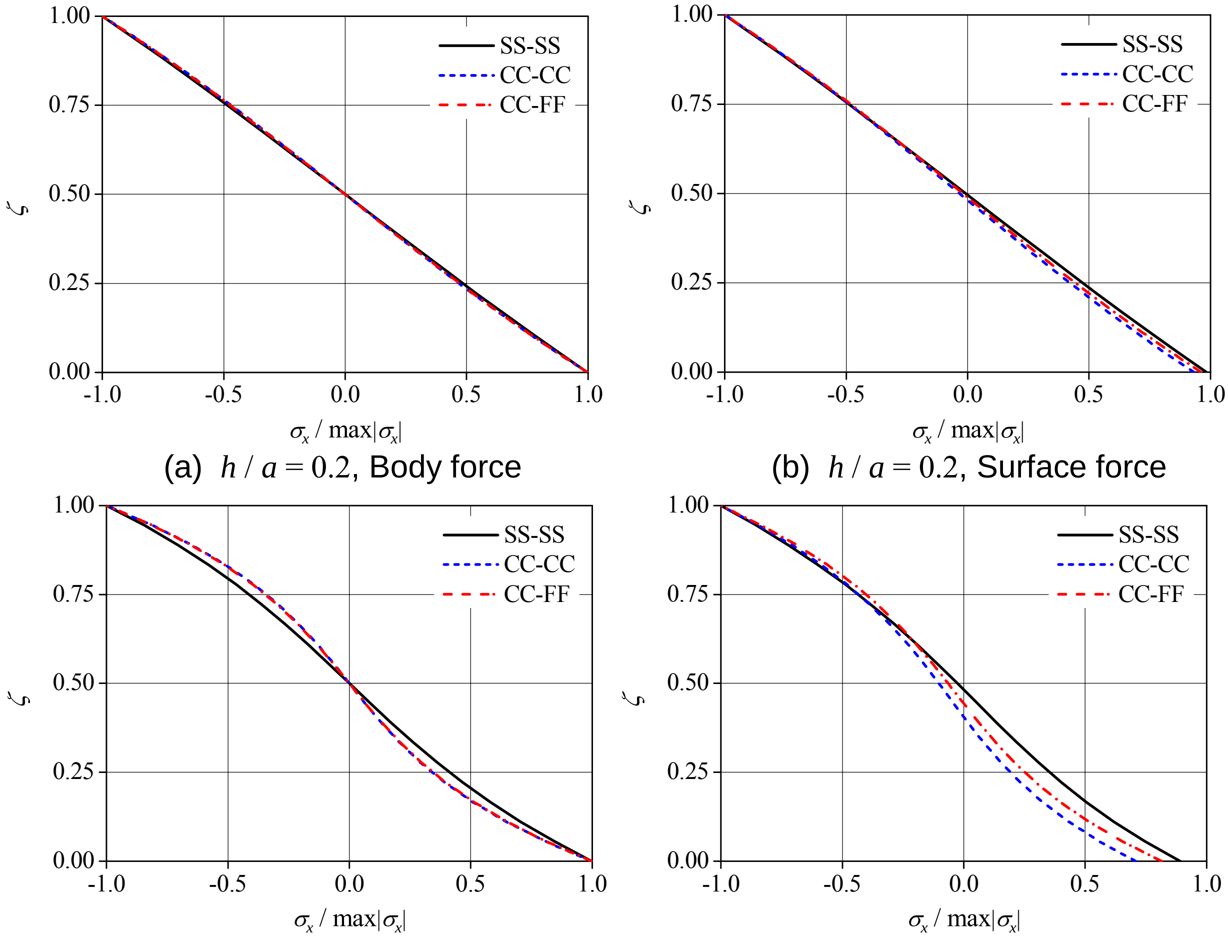
(a) $h / a = 0.2$, Body force (b) $h / a = 0.2$, Surface force

(c) $h / a = 0.5$, Body force (d) $h / a = 0.5$, Surface force

Figure 5. The effects of thickness-to-length ratio, boundary condition and loading condition on distribution of the in-plane normal stress of square thick plates on elastic foundations.

Figure 5 depict the effects of thickness-to-length ratio, boundary condition and loading condition on distribution of the in-plane normal stress σ_x / q_Γ ($\Gamma = Z, 0$) in the thickness direction of square plates on elastic foundations. The non-dimensional foundation parameter Θ is set as 10^{-2}. The thickness-to-length ratio h / a is used to be 0.2 and 0.5.

From Figure 5, the stress distributions in the thickness direction of moderately thick plates are linearly and anti-symmetric distributions with respect to middle surface ($\zeta =$ 0.5) regardless of boundary condition and transverse loading condition. However, the stress distributions in the thickness direction of thick plates are received to the effect of transverse loading condition. The stress distribution of thick plates on elastic foundation under the body force is non-linearly and anti-symmetrical with respect to middle surface ($\zeta = 0.5$). On the other hand, the stress distribution of thick plates on elastic foundation subjected to the surface force is not anti-symmetrical with respect to



(a) $h / a = 0.2$, Body force

(b) $h / a = 0.2$, Surface force

(c) $h / a = 0.5$, Body force

(d) $h / a = 0.5$, Surface force

Figure 6. The effects of thickness-to-length ratio, boundary condition and loading condition on distribution of the normalized in-plane normal stress of square thick plates on elastic foundations.

middle surface ($\zeta = 0.5$), and the neutral surface of normal stress for CC-FF plate and CC-CC plate has been moved down from the middle surface ($\zeta = 0.5$). Based on Figure 5, the distribution of the normalized in-plane normal stress σ_x / q_Γ (Γ = Z, 0) in the thickness direction of square plates on elastic foundations are shown in Figure 6. It is seen that the normalized stress distributions of moderately thick plates are not received to the effects of transverse loading conditions and boundary conditions. However, with increasing thickness, these effects are appeared to range from bottom surface ($\zeta = 0$) to middle surface ($\zeta = 0.5$).

4. COCLUSIONS

The 3-D stress and energy analysis of rectangular thick plates resting on elastic foundations with arbitrary boundary conditions has been presented. The thick plate is subjected to a body force and a surface force. The analysis is based on the 3-D theory of elasticity, and the elastic foundation is described by the Winkler model. The governing equation is formulated by the B–spline Ritz method. To demonstrate the convergence and accuracy of the present method, several examples are solved, and the results were compared with the analytical solutions and the finite element solutions. Moreover, the effects of thickness-to-length ratio, non-dimensional foundation parameter, transverse loading condition and boundary condition on the displacements and stresses of square thick plates resting on elastic foundations were clarified. As the results, the following important conclusions were obtained.

(1) Rapid, stable convergence and high accuracy were obtained by the B-spline method.
(2) The elastic foundation of range of $\Theta > 10^2$ is a rigid foundation, range of $10^{-2} \leq \Theta \leq 10^{2}$ is an elastic foundation, and range of $\Theta < 10^{-2}$ is same as the results of plate without foundation. Therefore, range of $\Theta < 10^{-2}$, the effect of the foundation can be neglected in the analysis.
(3) By the effects of transverse shear deformation and the surface force acting locally, distributions of normal stress in the thickness direction for thick plate become non-linearly, and the neutral surface of normal stress has been moved down.
(4) The normalized stress distributions of moderately thick plates are not received to the effects of transverse loading conditions and boundary conditions. However, with increasing thickness, these effects are appeared to range from bottom surface to middle surface.

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