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Isotropic Damage Model of Kachanov: Theoretical Formulation and Numerical Simulation

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ABSTRACT

In this paper, the isotropic damage model of Kachanov is studied under small deformation theory. From the standard damage model, the Kachanov damage model is carried out in the same manner. The Kachanov damage model is capable of presenting different phases of material behavior and easily implemented via the numerical method. Several numerical examples are provided to illustrate a very satisfying performance of the Kachanov damage model in 2D and 3D configurations.

1. INTRODUCTION

In the plasticity model, the elastic response is assumed to remain the same in loading and unloading. However an inelastic deformation pertains to cracks can modify the elastic response in unloading phase which are indeed observed in loading/unloading cycles of brittle materials, such as ceramics, glass or concrete. In order to reproduce this kind of inelastic behavior, the damage model is proposed by several researchers (e.g. Lematre and Chaboche 1988). One of the famous models is introduced by Kachanov in 1958, where the cracking phenomena is developed by a continuum mechanics model. In this model, a single internal variable of damage is exploited. For the intact material with no cracks, this internal variable of damage takes a value of zero, meanwhile it can further increase to 1 to present a fully damaged material. In 2D and 3D settings, the compliance tensor **D** of a damage state is selected as the internal damage variable. Hence the constitutive relation for a damaged state is then rewritten. The paper outline is as follows. In section 2, the theoretical formulation of Kachanov damage model subjected to small strain theory is presented. In section 3, several simulations are programmed in FEAP v8.4 to illustrate the behavior of material embedded with Kachanov damage model.

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2. THEORETICAL FORMULATION OF KACHANOV DAMAGE MODEL

The construction of 3D Kachanov damage model is briefly presented, see Ibrahimbegovic (2009). The internal damage variable for 3D damage model is chosen as the fourth order compliance tensor **D**, which is equal to the inverse of elasticity tensor **C** at the intact condition. The compliance tensor components are modified by the damage process, in order to represent the cracking produced by different damage mechanisms. All cracks are assumed to be inactive in unloading, hence

$$\nabla^s \mathbf{u} =: \boldsymbol{\epsilon} = \boldsymbol{\mathcal{D}} \boldsymbol{\sigma} \tag{1}$$

The complementary energy potential is defined correspondingly as follows.

$$\chi(\boldsymbol{\sigma}, \boldsymbol{\mathcal{D}}, \zeta) := \frac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\mathcal{D}} \boldsymbol{\sigma} - \Xi(\zeta) \tag{2}$$

The hardening potential energy is $\Xi(\zeta) = \frac{1}{2} q\zeta$. The damage criterion for standard model is governed by yield function Φ(**σ**,q).

$$0 \geq \phi(\boldsymbol{\sigma}, q) := \underbrace{(\boldsymbol{\sigma} \cdot \boldsymbol{\mathcal{D}}^e \boldsymbol{\sigma})^{1/2}}_{\|\boldsymbol{\sigma}\|_{\mathcal{D}^e}} - \frac{1}{\sqrt{E}} (\sigma_f - q) \tag{3}$$

where σ_f is stress limit of fracture, the function q(ζ)=-Kζ controls the evolution of the damage threshold via the constant hardening modulus K of prescribed material. The elastic regime is corresponding to negative value of the damage function Φ(σ,q)<0 where internal variable and compliance tensor remain unchanged. By contrast, a zero value of the damage function denotes the presence of damage evolution. This produces the damage dissipation, which satisfies the second thermodynamics principle.

$$\begin{aligned} 0 \leq D^d &:= \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\epsilon}} + \tfrac{\partial}{\partial t}[\chi(\boldsymbol{\sigma}, \boldsymbol{\mathcal{D}}, q) - \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}] \\ &= \dot{\boldsymbol{\sigma}}(-\boldsymbol{\epsilon} + \boldsymbol{\mathcal{D}} \boldsymbol{\sigma}) + \tfrac{1}{2} \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\mathcal{D}}} \boldsymbol{\sigma} + q\dot{\zeta} \; ; \; q = -\tfrac{d\Xi}{d\zeta} \end{aligned} \tag{4}$$

In an elastic process, with no change of internal variables and zero damage dissipation, the above equation confirms the constitutive equation as follows.

$$\left. \begin{array}{l} \dot{\boldsymbol{\mathcal{D}}} = \mathbf{0} \\ \dot{\zeta} = 0 \\ D^d = 0 \end{array} \right\} \implies \boldsymbol{\epsilon} = \boldsymbol{\mathcal{D}} \boldsymbol{\sigma} \tag{5}$$

In an inelastic process, the same constitutive equation is still validated hence a reduced form of the damage dissipation is revealed.

$$0 < D^d := \frac{1}{2} \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\mathcal{D}}} \boldsymbol{\sigma} + q\dot{\zeta} \tag{6}$$

In all admissible candidates, the solution of stress maximizes the damage dissipation in Eq. (6) following the principle of maximum damage dissipation. Thus this problem is transformed into a constrained minimization problem. By means of the Lagrange multiplier method, the unconstrained minimization problem is written as follows.

$$\begin{aligned} &\min_{\phi(\sigma,q)\leq 0}[-D^d] \implies \max_{\dot{\gamma}\geq 0} \min_{\forall(\sigma,q)} L^d(\boldsymbol{\sigma}, q, \dot{\gamma}) \\ &L^d(\boldsymbol{\sigma}, q, \dot{\gamma}) := -D^d(\boldsymbol{\sigma}, q) + \dot{\gamma}\phi(\boldsymbol{\sigma}, q) \end{aligned} \tag{7}$$

By exploiting the last result, it is possible to provide the explicit form of the evolution equations from the corresponding Kuhn-Tucker optimal conditions for minimization problem in Eqs. (8) and (9),

$$\left.\begin{aligned} 0 &= -\boldsymbol{\sigma} \cdot \dot{\boldsymbol{\mathcal{D}}}\, d\boldsymbol{\sigma} + \dot{\gamma}\frac{\partial \phi}{\partial \boldsymbol{\sigma}} \cdot d\boldsymbol{\sigma} \\ &= -\boldsymbol{\sigma} \cdot \dot{\boldsymbol{\mathcal{D}}}\, d\boldsymbol{\sigma} + \dot{\gamma}\frac{\boldsymbol{\mathcal{D}}^e \boldsymbol{\sigma}}{\|\boldsymbol{\sigma}\|_{D^e}}\, d\boldsymbol{\sigma} \end{aligned}\right\} \implies \dot{\boldsymbol{\mathcal{D}}} = \frac{\dot{\gamma}}{\|\boldsymbol{\sigma}\|_{D^e}} \boldsymbol{\mathcal{D}}^e \tag{8}$$

These equations are accompanied by loading/unloading conditions, which are also obtained as a part of the Kuhn-Tucker optimal conditions

$$\dot{\gamma} \geq 0 \; ; \; \phi \leq 0 \; ; \; \dot{\gamma}\phi = 0 \tag{9}$$

A new definition of the damage multiplier is introduced leading to a simple evolution equation for damage internal variable

$$\dot{\bar{\mu}} = \frac{\dot{\gamma}}{\|\boldsymbol{\sigma}\|_{D^e}} \tag{10}$$

The corresponding evolution of the compliance tensor is computed from the intact state.

$$\dot{\boldsymbol{\mathcal{D}}} = \dot{\bar{\mu}}\boldsymbol{\mathcal{D}}^e \implies \boldsymbol{\mathcal{D}}(t) = [1 + \bar{\mu}(t)]\boldsymbol{\mathcal{D}}^e \; ; \; \mu(0) = 0 \tag{11}$$

Introducing a definition of the damage variable d∈[0,1], the stress is now computed as follows.

$$d = \frac{\bar{\mu}}{1+\bar{\mu}} \, ; \, d \in [0,1] \implies \boldsymbol{\mathcal{D}}^{-1} = (1-d)\boldsymbol{\mathcal{C}} \Leftrightarrow \boldsymbol{\sigma} = (1-d)\boldsymbol{\mathcal{C}}\boldsymbol{\epsilon} \tag{12}$$

The value of the Lagrange multiplier is consequently revealed from Eq. (8).

$$\dot{\gamma} = \frac{\frac{\partial \phi}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\mathcal{D}}^{-1} \dot{\boldsymbol{\epsilon}}}{\frac{\partial \phi}{\partial \boldsymbol{\sigma}} \cdot \boldsymbol{\mathcal{D}}^{-1} \frac{\partial \phi}{\partial \boldsymbol{\sigma}} + \frac{d^2 \Xi}{d\varsigma^2}} \tag{13}$$

The constitutive equation for the stress and strain rate is derived, hence reveal the form of tangent elastodamage modulus for the continuum problem.

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{C}}^{ed}\dot{\boldsymbol{\epsilon}} \; ; \; \boldsymbol{\mathcal{C}}^{ed} = \begin{cases} \boldsymbol{\mathcal{D}}^{-1} \; ; \; \dot{\gamma} = 0 \\ \boldsymbol{\mathcal{D}}^{-1} - \frac{1}{\left(\frac{\partial\phi}{\partial\boldsymbol{\sigma}}\cdot\boldsymbol{\mathcal{D}}^{-1}\frac{\partial\phi}{\partial\boldsymbol{\sigma}} + \frac{d^2\Xi}{d\varsigma^2}\right)}\boldsymbol{\mathcal{D}}^{-1}\frac{\partial\phi}{\partial\boldsymbol{\sigma}} \otimes \boldsymbol{\mathcal{D}}^{-1}\frac{\partial\phi}{\partial\boldsymbol{\sigma}} \end{cases} \tag{14}$$

The standard finite element approach is employed to discretize the physical domain and construct corresponding discrete governing equations. Consequently, the stiffness matrix is assembled over entire domain of structure and the conventional equation $f^{int} = f^{ext}$ is established in conventional manner. Details on finite element approach can be found in Zienkiewicz et al (2005).

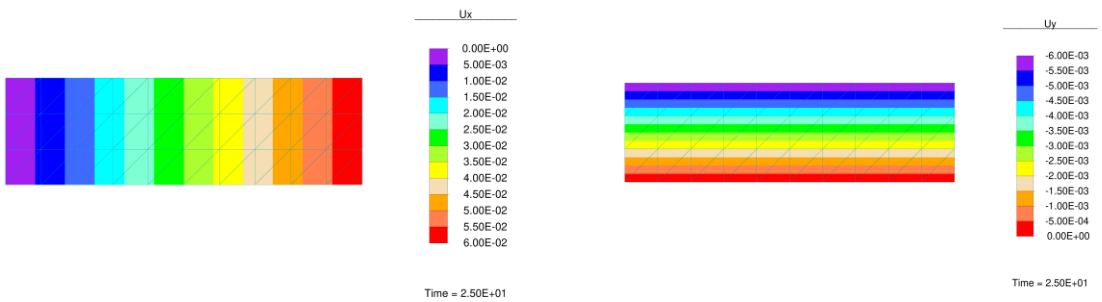
$$\begin{aligned} &\mathop{\mathrm{A}}_{e=1}^{n_{elem}} \mathbf{K}_{n+1}^{e,(i)}\mathbf{u}_{n+1}^{e,(i)} = \mathbf{f}_{n+1}^{ext} \\ &\mathbf{K}_{n+1}^{e,(i)} = \int_{\Omega^e} \boldsymbol{B}_{n+1}^{e,(i),T}\boldsymbol{C}_{n+1}^{ed,(i)}\boldsymbol{B}_{n+1}^{e,(i)}dx \\ &\mathbf{d}_{n+1}^{(i+1)} = \mathbf{d}_{n+1}^{(i)} + \mathbf{u}_{n+1}^{(i)} \end{aligned} \tag{15}$$

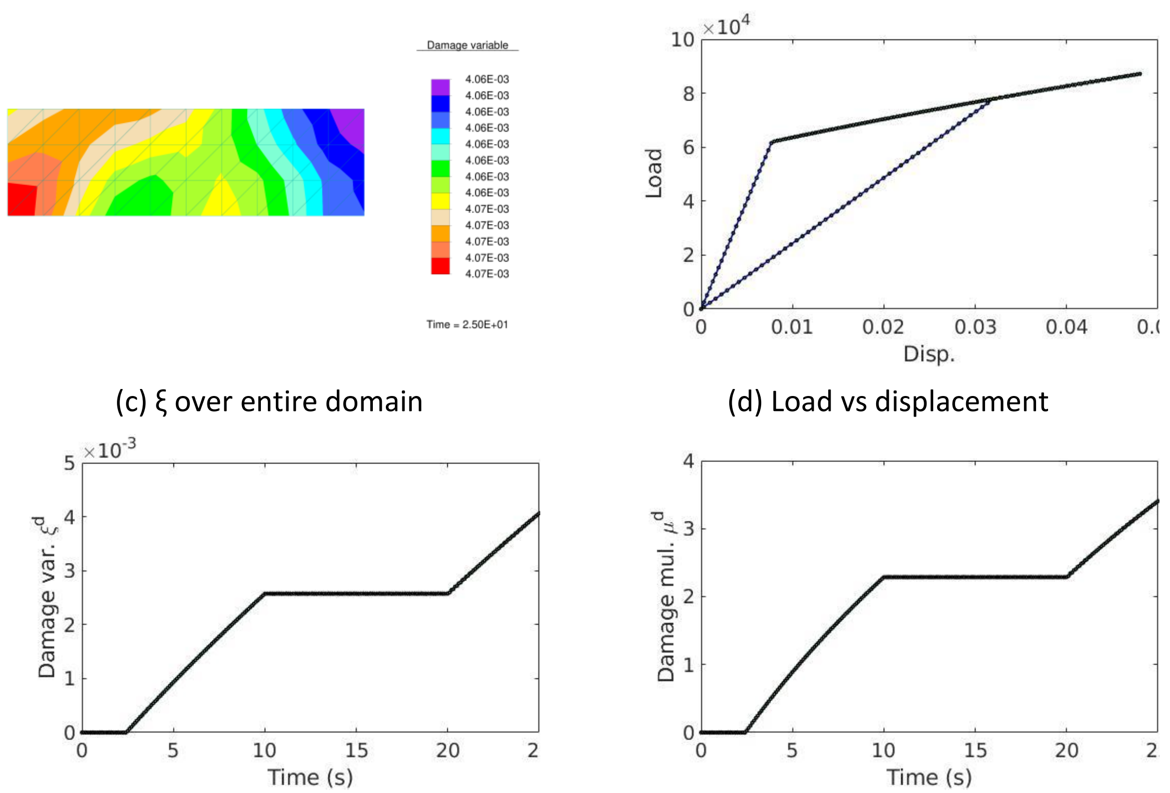
3. REPRESENTATIVE NUMERICAL EXAMPLES

The Kachanov damage model is programmed under quasi-static analysis in FEAP v8.4 (Taylor 2015). In the 2D case, the isoparametric constant stress/strain triangle element is employed, while the isoparametric linear tetrahedron element is utilized for 3D case. The material properties are listed as: elastic modulus E = 200e5, poisson's ratio ν = 0.25, stress limit of fracture σ_f = 30e3 and hardening modulus K = 200e4. The imposed displacement at the right end in 2D case or at the top of specimen in 3D case. The loading diagram of imposed displacement is plotted within 250 time steps (dt=0.1s) in Fig. 2(h). These simulations converge with 2 iterations for elastic regime and 5 iterations for inelastic regime (total residual energy under 1e-25).

3.1 Simple tension test in 2D

The test specimen is a 10x3 rectangular bar. Each left and right end is clamped by simple rollers. The displacement field along x and y direction is shown in Fig. 1(a-b). The value of damage variable is around 4.06e-3 over entire domain at the final step as in Fig. 1(c). The internal damage variable is frozen when the specimen is in elastic phase or unloading regime. Meanwhile, the internal damage variable increases along with time when the specimen is under loading regime as in Fig. 1(d-f).





(a) Displacement u_x (b) Displacement u_y

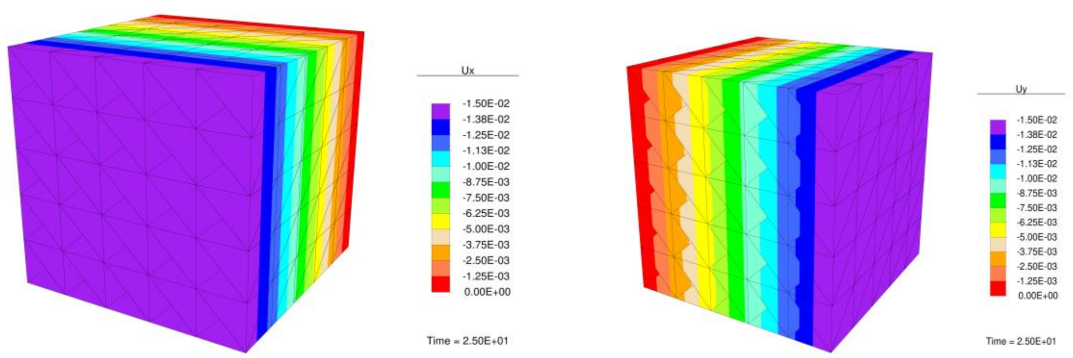
(c) ξ over entire domain (d) Load vs displacement

(e) Evolution of ξ at element 18 (f) Evolution of μ at element 18

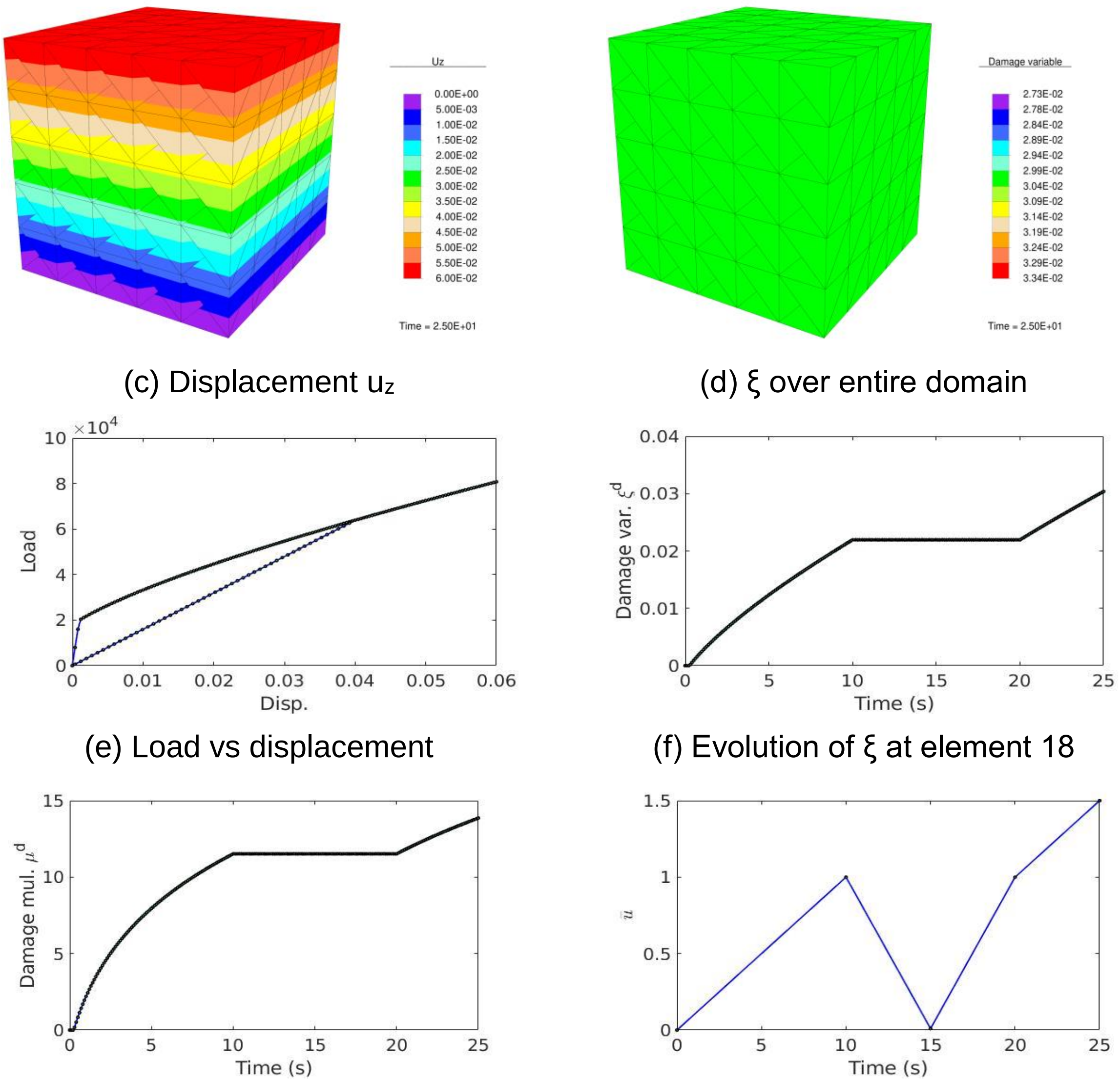
Fig. 1 Simple tension test in 2D - Kachanov damage model

3.2 Simple tension test in 3D

The test specimen is a 1x1x1 cube. In each direction, it is clamped by simple rollers. The displacement field along each direction is shown in Fig. 2(a-c). The value of damage variable is nearly constant around 3e-2 over entire domain at the final step as in Fig. 2(d). The internal damage variable is frozen when the specimen is in elastic phase or unloading regime. Meanwhile, the internal damage variable increases along with time when the specimen is under loading regime as in Fig. 2(e-g).



(a) Displacement u_x (b) Displacement u_y



(c) Displacement u_z (d) ξ over entire domain

(e) Load vs displacement (f) Evolution of ξ at element 18

(g) Evolution of μ at element 18 (h) Loading diagram $(\bar{u})$

Fig. 2 Simple tension test in 3D - Kachanov damage model

4. CONCLUSION

The isotropic damage model of Kachanov is reviewed under small deformation theory. The development of Kachanov model is established on the standard damage model. Several phases of material damage are presented by Kachanov damage model in the numerical examples. The results show a good performance of Kachanov damage model in both two-dimensional and three-dimensional settings.

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