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3-D Transient Analysis of Elastic Wave Scattering by a Crack Using a Convolution Quadrature Based Boundary Element Method

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ABSTRACT

The ultrasonic nondestructive evaluation (NDE) is one of the most widely used NDE methods.In the ultrasonic NDE, scattered wave forms are used for the evaluation of defects in materials. Therefore, numerical methods for elastic wave scattering are particularly useful for the quantification of performance of the ultrasonic NDE techniques. In this research, a convolution quadrature based boundary element method is developed for 3-D transient analysis of elastic wave scattering by a crack.

1. INTRODUCTION

The ultrasonic nondestructive evaluation (NDE) is one of the most widely used NDE methods (Achenbach, 1984). In the ultrasonic NDE, scattered wave forms are used for the evaluation of defects in materials. Therefore, numerical methods for elastic wave scattering are particularly useful for the quantification of performance of the ultrasonic NDE techniques.

A time-domain boundary element method (TD-BEM) is effective numerical approach for ultrasonic NDE simulation because it can deal with infinite or half space region without any modifications. The TD-BEM has significant advantages for mathematical modeling of cracks that only the surfaces of the part need to be meshed. Therefore, the TD-BEM has been applied to the analysis of elastic wave scattering by cracks for the past few decades.However,theconventional TD-BEM sometimes suffers from numerical instability in thetime-marching procedure.

In this research,an innovative TD-BEM, which is called a CQ-BEM, is developed for the analysis of elastic wave scattering by a crack. A convolution quadrature method (CQM) developed by (Lubich, 1988) is applied to the TD-BEM to improve the numerical stability. The CQM approximates convolution integrals of time-domain boundary integral equations by a quadrature formula numerically and stably. The CQ-BEM requires Laplace-domain fundamental solutions. This fact makes it possible to easily for-

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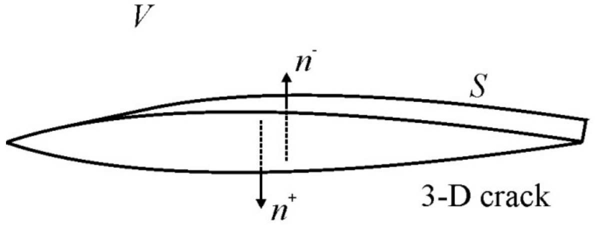


Fig.1 A 3-D crack in an infinite and homogeneous elastic solid V.

mulate the new TD-BEM approach.In fact, Saitoh et al. (Saitoh et al.,2009) formulated a CQ-BEM for 2-D viscoelastic wave propagation and it was accelerated by the fast multipole method (FMM). Furukawa et al (Furukawa et al., 2012) solved a problem of an elastic wave scattering by a crack using the Galerkin CQ-BEM. In this paper, a CQ-BEM formulation for 3-D elastic wave scattering by a crack is presented. The scattering problems of an incidentplane wave by a crack in an isotropic material are solved to validate the proposed method.

2. REGULARIZED HYPERSINGULAR INTEGRAL EQUATION

In this paper, a Latin suffix takes values 1, 2, and 3, unless otherwise stated. The summation convention is valid for repeated indices throughout the paper.

Let us consider an elastic wave scattering by a crack S in an infinite and homogeneous elastic solid$V$, as shown in Fig.1.The equation of motion at point$\boldsymbol{x}$and time t are given as follows:

$$\sigma_{ij,j}(\boldsymbol{x},t)+\rho p_i(\boldsymbol{x},t)=\rho\ddot{u}_i(\boldsymbol{x},t) \tag{1}$$

where$u_i(\boldsymbol{x},t)$, $\sigma_{ij}(\boldsymbol{x},t)$, and $p_i(\boldsymbol{x},t)$ represent the displacement, stress, and body force component, respectively. Thedot notation($\dot{}$)and the variable ($)_{,i}$denote partial derivative with respect to time t and space x_i , respectively. In addition, ρ is the density of the elastic solid V.The stress-strain relation is given by

$$\sigma_{ij}(\boldsymbol{x},t)=C_{ijkl}u_{k,l}(\boldsymbol{x},t) \tag{2}$$

whereC_{ijkl} is the elastic tensor, given by $C_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$.In addition, δ_{ij} is the Kronecker delta, and λ and μ are Lamé constants. The hypersingular integral equation is written as follows:

$$t_p^{in}(\boldsymbol{x},t)=-C_{pqli}n_q^+(\boldsymbol{x})\mathrm{p.f.}\int_S C_{iknj}\frac{\partial}{\partial x_l\partial y_n}U_{mj}(\boldsymbol{x},\boldsymbol{y},t)n_k^+(\boldsymbol{y})*\phi_i(\boldsymbol{y},t)\,dS,\boldsymbol{x}\in V \tag{3}$$

wherep.f. indicates the finite part of the integral and * is the convolution integral with respect to time t. $U_{mj}(\boldsymbol{x},\boldsymbol{y},t)$is the fundamental solution for 3-D elastodynamics, n_k^+ represents the outward unit normal vector component, and $\phi_i(\boldsymbol{x})$ shows the crack opening displacement. In addition, $t_p^{in}(\boldsymbol{x},t)$ is the traction component of an incident wave$u_p^{in}(\boldsymbol{x},t)$. The hyper singular integral equationdefined in Eq. (3) can be regularized as

$$t_p^{in}(\boldsymbol{x},t) = C_{pqkl} n_q^+(\boldsymbol{x}) \left[e_{rst} e_{rlj} \mathrm{p.v.} \int_S C_{ijnm} U_{nk,m}(\boldsymbol{x},\boldsymbol{y},t) * \phi_{i,t}(\boldsymbol{y},t) n_s^+(\boldsymbol{y}) dS + \rho \int_S U_{ik}(\boldsymbol{x},\boldsymbol{y},t) * \ddot{\phi}_i(\boldsymbol{y},t) n_l^+(\boldsymbol{y}) dS \right] \quad (4)$$

wherep.v. indicates the Cauchy principal value integral and e_{rst} is the permutation symbol. In addition, $\sigma_{ijk}(\boldsymbol{x},\boldsymbol{y},t) = C_{ijnl} U_{nk,l}(\boldsymbol{x},\boldsymbol{y},t)$. Normally, the time-domain boundary integral equation (4) is discretized by the time-stepping scheme. However, the well known time-stepping scheme sometimes produces numerical errors if we use small time step sizeΔt. To overcome the difficulty, the CQM is utilized for the time discretization of the boundary integral equation.

3. CONVOLUTION QUADRATURE METHOD

The CQM, firstproposed by Lubich, approximates the convolution $f * g(t)$ by a discrete convolution using theLaplace transform of the time-dependent function $f(\tau - t)$. In general, the convolution integralis approximated by the CQM as follows:

$$f * g(n\Delta t) \simeq \sum_{j=0}^{n} \omega_{n-j}(\Delta t) g(j\Delta t), \; n = 0,1,\dots,N \quad (5)$$

where$t$ is divided into Nequal steps,Δt, and $\omega_j(\Delta t)$is the quadrature weight. The quadrature weightis determined with the Laplace transform of the original time-dependent function $f$and given as follows:

$$\omega_n(\Delta t) \simeq \frac{R^{-n}}{L} \sum_{l=0}^{L-1} \hat{f}\left(\frac{\gamma(\zeta_l)}{\Delta t}\right) e^{-\frac{2\pi \mathrm{i} nl}{L}} \quad (6)$$

where$\hat{f}$is the Laplace transform ofthe function f, i is the imaginary unit, and $\gamma(\zeta)$is the quotient of the generating polynomials of linear multistep method given by $\gamma(\zeta) = \sum_{i=1}^{k} (1-\zeta)^i / i$using backward differential formulas (BDF) and ζ_l is given by $\zeta_l = Re^{2\pi \mathrm{i} l/L}$. Inaddition, R is the radius of a circle in the analyticity domain of$\hat{f}$ and ϵ is the error of the numerical calculation of Eq.(6). Normally, the parameter R is calculated by

$$R^L = \sqrt{\epsilon}. \qquad (7)$$

The CQM parameter L is set as $L = N$ to accelerate the calculation of influence functions by using FFT, as is mentioned later.

4. DISCRETIZATION OF REGULARIZED INTEGRAL EQUATIONS USING THE CQM

Discretizing the regularized hypersingular integral equations (4) using a piecewise constant approximation of the unknown crack opening displacement $\phi_i(\boldsymbol{y}, t)$ and using the CQMfor the convolutions of Eq. (4) yield the discretized boundary integral equation as follows:

$$t_p^{in}(\boldsymbol{x}, n\Delta t) = \sum_{\alpha=1}^{M} \sum_{k=1}^{n} \left[A_{pi}^{n-k}(\boldsymbol{x}, \boldsymbol{y}^{\boldsymbol{\alpha}}) + B_{pi}^{n-k}\ (\boldsymbol{x}, \boldsymbol{y}^{\boldsymbol{\alpha}})\right] \phi_i(\boldsymbol{y}^{\boldsymbol{\alpha}}, n\Delta t) \qquad (8)$$

where M is the number of discretized boundary elements. In addition, $A_{pi}^m(\boldsymbol{x}, \boldsymbol{y})$ and $B_{pi}^m(\boldsymbol{x}, \boldsymbol{y})$ are the influence functions, which are defined by

$$A_{pi}^m(\boldsymbol{x}, \boldsymbol{y}) = \rho C_{pqkl} n_q(\boldsymbol{x}) n_l^+(\boldsymbol{y}) \frac{R^{-m}}{L} \sum_{l=0}^{L-1} \left[\int_S \left(\frac{\gamma(\zeta_l)}{\Delta t}\right)^2 \widehat{U}_{ik}(\boldsymbol{x}, \boldsymbol{y}, s_l) dS_y\right] e^{-\frac{2\pi iml}{L}} \qquad (9)$$

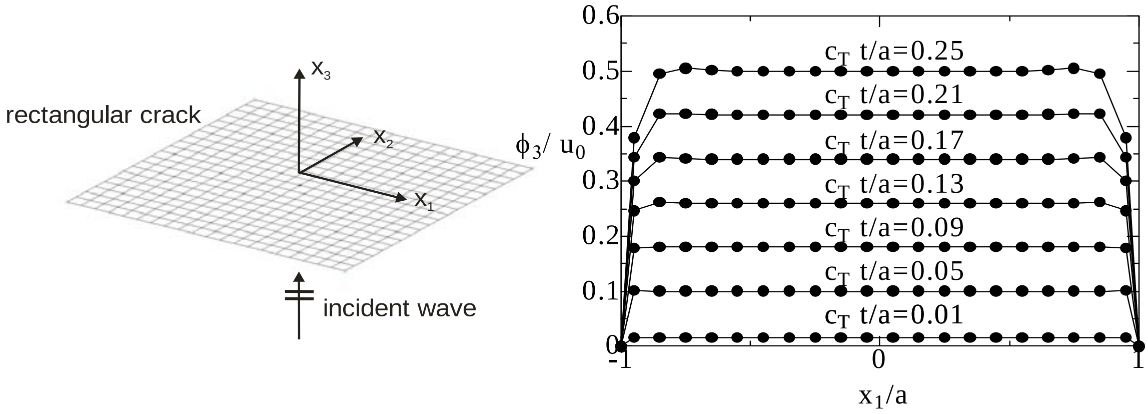
$$B_{pi}^m(\boldsymbol{x}, \boldsymbol{y}) = C_{pqkl} C_{ijnl} n_q^+(\boldsymbol{x}) e_{rlj} \frac{R^{-m}}{L} \sum_{l=0}^{L-1} \left[\int_{\partial S} \frac{\partial \widehat{U}_{nk}}{\partial y_l}(\boldsymbol{x}, \boldsymbol{y}, s_l) ds_r\right] e^{-\frac{2\pi iml}{L}} \qquad (10)$$

where $\widehat{U}_{ik}(\boldsymbol{x}, \boldsymbol{y}, s)$ is the Laplace-domain fundamental solution for 3-D elastodynamics. s_l denotes the Laplace parameter given by $s_l = \gamma(\zeta_l)/\Delta t$ and ds_r shows sides of each boundary element S. Equations (9) and (10) are identical to the Fourier transforms. Therefore, these equations can be rapidly evaluated using the FFT algorithm.Equation (8) can be solved with initial conditions from the first time step.

5. NUMERICAL RESULTS

3-D transient analysis of elastic wave scattering by a crack is implemented by the proposed method. In this calculation, the error parameter $\epsilon = 1.0 \times 10^{-16}$ is used for Eq. (7). The time increment $c_T \Delta t/a$ =0.01 and the Poisson's ratio $\nu = 0.25$. The incident wave is given as follows:

$$u_i^{in}(\boldsymbol{x}, t) = u^0 d_i^p \frac{a}{c_T}\left(t - \frac{\boldsymbol{d} \cdot \boldsymbol{x}}{c_L}\right) H\left[t - \frac{\boldsymbol{d} \cdot \boldsymbol{x}}{c_L}\right] \qquad (11)$$



(a) (b)

Fig.2 Elastic wave scattering by a rectangular crack. (a) rectangular crack model (b) time variations of crack opening displacements on x_1 axis .

whereu^0 is the displacement amplitude, $H[\cdot]$ is the step-function, and T is the waveperiod. In addition, d_i^p is the polarization vector component and $\boldsymbol{d}$ is the propagation vector.$c_L$is the P-wave velocity.

As a first example, the crack opening displacements of the rectangular crack with the length of $2a$, as shown in Fig.2(a), are calculated using the proposed CQ-BEM. The rectangular crack is discretized into 400 elements and the total number of time step is $= 32$. Figure 2(b) shows time variations of the crack opening displacementsϕ_3 on x_1 axis in Fig.2 (a). In this case, N_{iw}=1 and $\boldsymbol{d^p} = \boldsymbol{d} = (1,0,0)$ are considered in Eq. (11). At $c_T t/a = 0.0$, ϕ_3/u_0 is identical to zero everywhere. Dimensionless crack opening displacements ϕ_3/u_0 show large values as dimensionless time $c_T t/a$ increses. As expected, ϕ_3/u_0 is almost zero on both sides of the crack tips.The numerical results obtained by our proposed method are good agreement with those obtained by the classical time-domain BEM (Zhang, 1993).

Next, elastic wave scattering by a penny shaped crack with radius a subjected to the incident wave defined by

$$u_i^{in}(\boldsymbol{x}, t) = u^0 d_i^p H\left[t - \frac{\boldsymbol{d}\cdot\boldsymbol{x}}{c_L}\right] H\left[N_{iw}T - \left(t - \frac{\boldsymbol{d}\cdot\boldsymbol{x}}{c_L}\right)\right] \sin\left\{\frac{2\pi}{T}\left(t - \frac{\boldsymbol{d}\cdot\boldsymbol{x}}{c_L}\right)\right\} \qquad (11)$$

is analyzed using the proposed method.In this case, the variable N_{iw} is set to be $N_{iw} = 5$. The number of boundary element is $M = 96$, and the total time step is $n = L = 256$.The propagating and polarization vectors are given by $\boldsymbol{d^p} = \boldsymbol{d} = (1,0,0)$ **.** Figures 3(b) and (c) show elastic wave fields $|\boldsymbol{u}|/u_0$ in x_1-x_3 plane around the crack at $c_T t/a = 0.5$ and $c_T t/a = 1.9$, respectively.We can see that the scattered waves are generated by the interaction between the incident wave and the crack.

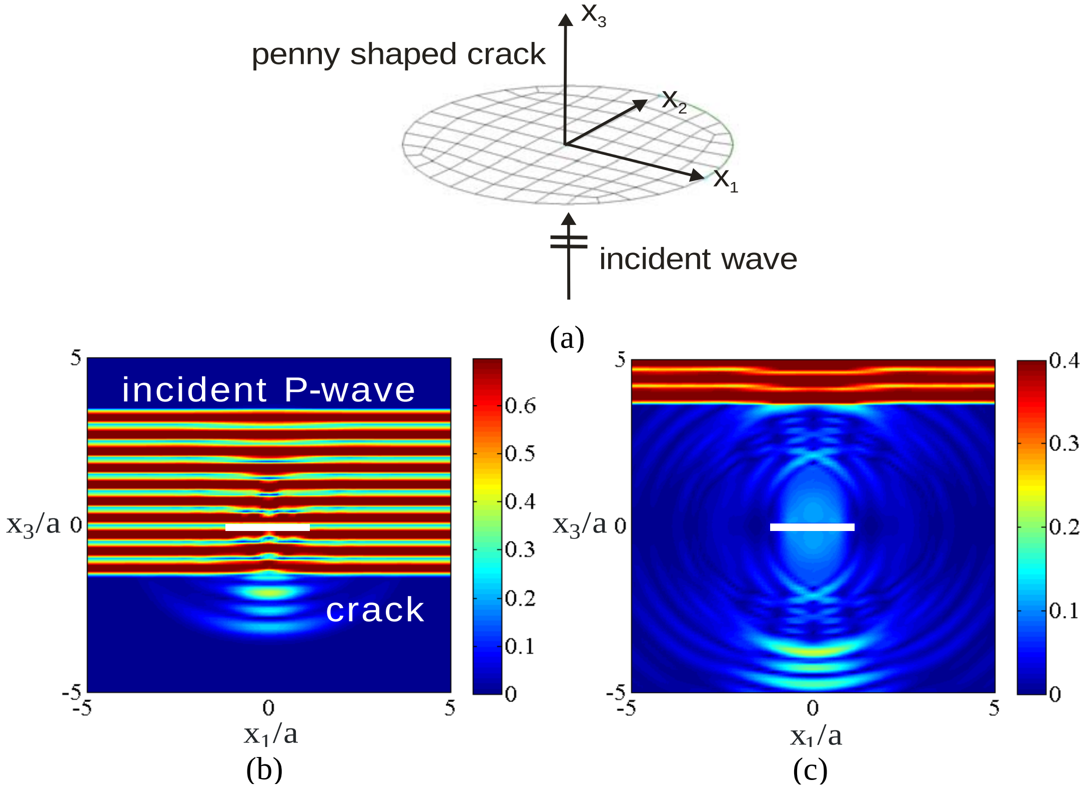


Fig.3Elasticwave scattering by a penny shaped crack. (a) penny shaped crack model(b) elastic wave field around the crack at $c_T t/\mathrm{a} = 0.5$ (c) $c_T t/\mathrm{a} = 1.9$.

6. CONCLUSIONS

In this research, the CQ-BEM was developed for 3-D transient analysis of elastic wave scattering by a crack. The CQM was applied to time-discretization of the convolution integrals of the regularized hypersingular integral equations. The proposed method will be applied to the simulation of non-linear higher harmonic generation of ultrasonic waves in near future.

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