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Resonant behavior of railway track having random sleeper spacing

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ABSTRACT

The influence of randomness in sleeper spacing on the resonant behavior of a railway track is investigated. We consider railway track vibrations excited by a harmonic load. The track is modeled by a finite but sufficiently long rail supported by a number of sleepers. The influence of position of each sleeper on the pinned-pinned resonance mode is investigated through numerical analyses. Furthermore, the first- and second-order derivatives of the response with respect to the position of sleepers are evaluated. Based on these results, the availability of the perturbation method for the evaluation of the statistical descriptors such as the mean value and variance of the resonant response is also discussed under a probability density function of the sleeper spacing.

1. INTRODUCTION

A railway track composed of continuous welded rails and sleepers can be regarded as a periodic structure due to the discrete sleeper supports. Owing to this periodicity the band structure of wave modes propagating in the track is characterized by the existence of frequency ranges called stop bands or band gaps in which the propagation of bending waves along the rail is forbidden (Mead 1970, Hosking 2004). In particular, the standing wave modes locating at these band edges dominate the resonant response of the track. Therefore, it is essential for the understanding of dynamic behavior of a track to obtain the dispersion curves.

However, in general, railway tracks have non-uniform sleeper spacing. This irregularity may affect the vibration behavior of the track. For example, through numerical experiments Wu and Thompson (2000) found that the so-called pinned-pinned resonance mode which has nodes at each sleeper support is suppressed due to the scatter in the sleeper spacing. Heckl (1995) has studied the effect of random sleeper spacing from the viewpoint of noise reduction, and concluded that the randomization contributes to the reduction of rolling noise at 1200Hz or lower. The authors (Batjargal 2012) have attempted to optimize the sleeper arrangement based on an objective function defined by the wave energy transmission and accomplished the

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vibration reduction. Although, these papers have revealed beneficial features of the deviation from a periodicity, the introduction of disorder in a periodic structure might amplify the vibration due to the localization of wave energy (Duclos 2004, Li 2005). Nordborg (1998) has considered the irregularities in sleeper spacing and support stiffness, and found that the track vibration level can be increased at rather low frequencies.

While the qualitative relationship between the randomness in a track and its response has been deduced from numerical simulations, the dependence of the dynamic response upon individual varying parameters such as the position of a sleeper has never been clarified. Practically, the grasp of the sensitivity of vibration behavior to variations in certain stochastic properties will contribute to the technical progress in the design and maintenance of tracks.

Once the mechanical conditions including the irregularities are determined for a numerical model, dynamic response of the track can be determined readily. Nevertheless, the extraction of some statistical data such as the mean value and variance of the response will consume considerable computation time. Therefore, in order to save the computational effort, some mathematical approaches such as the perturbation method will be helpful for the evaluation of statistical parameters. Oscarsson (2002) has assessed the influence of stochastic properties on the train-track dynamic interaction based on the first-order perturbation technique. Since the first-order approximation cannot predict the relationship between the scatter and the mean value of the response, higher order estimation is needed for this purpose.

In this paper the influence of randomness in the sleeper spacing on the resonant behavior of a railway track is investigated, in the context of the vibration reduction. To achieve this, we analyze railway track vibrations excited by a harmonic load. In the numerical analyses, the track is modeled by a finite but sufficiently long rail supported by a number of sleepers. The influence of the position of each sleeper on the pinned-pinned resonance mode is examined through numerical simulations for tracks having different disordered sleeper distributions. From this result, the first- and second-order derivatives of the response amplitude with respect to the position of individual sleepers are evaluated by means of the finite deference method. Furthermore, direct simulations are carried out for tracks generated by a probability density function of the sleeper position, and the relationships between the variance of deviation in the sleeper location and the mean value and variance of the resonant amplitude are obtained. Based on these results, the availability of the perturbation method for the evaluation of the statistical descriptors is also discussed.

2. STATISTICAL RELATIONSHIP BETWEEN POSITION AND SPACING OF SLEEPERS

In this paper we formulate the influence of the disordered sleeper distribution on the resonant behavior in terms of the sleeper position. However, it may be rather practical to measure the sleeper spacing than the sleeper position. Consequently, in general, some statistical properties about the sleeper spacing will be obtained as primary data. Hence, this section is devoted to derive the statistical relationship between the position and the spacing of sleepers.

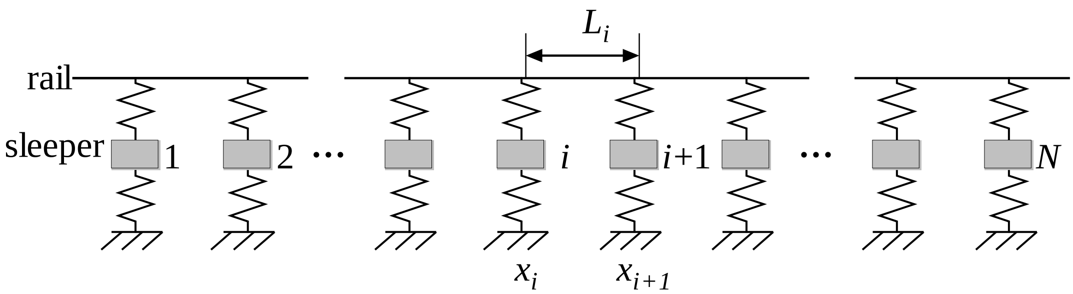

Fig. 1 Track model

2.1 Position and spacing of sleepers

Although a railway track with continuous welded rails can be modeled as an infinite structure, in this study a finite rail supported by *N* sleepers is considered. The sleepers are numbered from left to right as illustrated in Fig. 1. In the figure x_i is the position of the *i*th sleeper and L_i is the distance between the *i*th and *i*+1th sleepers, i.e.

$$L_i = x_{i+1} - x_i, \quad (i = 1, \cdots N-1). \tag{1}$$

The statistical relationship between the position and the spacing of sleepers will depend on the way in which the sleeper arrangement is made. We consider two ways of Case 1 and Case 2. In the former case, the location of the *i*+1th sleeper is determined based on the position of the *i*th sleeper sequentially. While in the latter case, all sleeper positions are determined simultaneously in a range of the track, so that a regular spacing can be realized.

2.2 Statistical relationship in Case 1

During the installation process, the position of the *i*+1th sleeper x_{i+1} is determined such that $L_i \approx L$. This procedure can be formulated by

$$x_{i+1} = x_i + L + \varepsilon_i, \tag{2}$$

where *L* is the average of sleeper spacing and ε_i is the deviation in the position of the *i*+1th sleeper. We assume that the mean value and the variance of ε_i are given by $E(\varepsilon_i)=0$ and $Var(\varepsilon_i)=\sigma_\varepsilon^2$ respectively, and the deviations of different sleepers are statistically independent, i.e. $Cov(\varepsilon_i, \varepsilon_j)=0$ for $i \neq j$.

The deviation of sleeper spacing μ_i is defined by

$$\mu_i = L_i - L. \tag{3}$$

From Eqs.(2) and (3), the following relation is obtained,

$$\varepsilon_i = \mu_i. \tag{4}$$

Therefore, $E(\mu_i)=0$, $Var(\mu_i)=\sigma_\varepsilon^2$, $Cov(\mu_i, \mu_j)=0$ for $i\neq j$.

Furthermore, Eq.(2) can be rewritten as

$$x_{i+1} = iL + \sum_k^i \varepsilon_k. \tag{5}$$

Since $E(\varepsilon_i)=0$ and $Cov(\varepsilon_i, \varepsilon_j)=0$ for $i\neq j$, Eq.(5) leads to the next relations,

$$E(x_{i+1}) = iL, \quad Var(x_{i+1}) = i\sigma_\varepsilon^2. \tag{6}$$

Notice that, in Case 1, the variance of the sleeper position increases proportionally to the sleeper number.

2.3 Statistical relationship in Case 2

In this case, x_i is given by

$$x_i = (i-1)L + \varepsilon_i, \tag{7}$$

where $E(\varepsilon_i)=0$, $Var(\varepsilon_i)=\sigma_\varepsilon^2$ and $Cov(\varepsilon_i, \varepsilon_j)=0$ for $i\neq j$, as in Case 1. From this equation, the mean value of x_i is obviously given by $E(x_i)=(i-1)L$.

Substituting Eq.(7) into Eq.(1), we can obtain the ith sleeper spacing as,

$$L_i = L - \varepsilon_i + \varepsilon_{i+1}. \tag{8}$$

Therefore, the deviation of sleeper spacing μ_i is given by

$$\mu_i = -\varepsilon_i + \varepsilon_{i+1}. \tag{9}$$

The mean value, variance and covariance of μ_i are thus obtained as

$$\begin{aligned} &E(\mu_i) = 0, \\ &Var(\mu_i) := \sigma_\mu^2 = 2\sigma_\varepsilon^2, \\ &Cov(\mu_i, \mu_j) = \begin{cases} -\sigma_\varepsilon^2 & (|i-j|=1), \\ 0 & (|i-j|>1). \end{cases} \end{aligned} \tag{10}$$

Consequently, the variance and covariance depend on the way of sleeper arrangement.

In general, at least in Japan, the sleeper spacing is specified by the number of sleepers installed in a certain distance. Since this condition corresponds to Case 2, the disorder in the sleeper spacing will be governed by Eq.(10). Therefore, the following discussion is conducted based on Case 2.

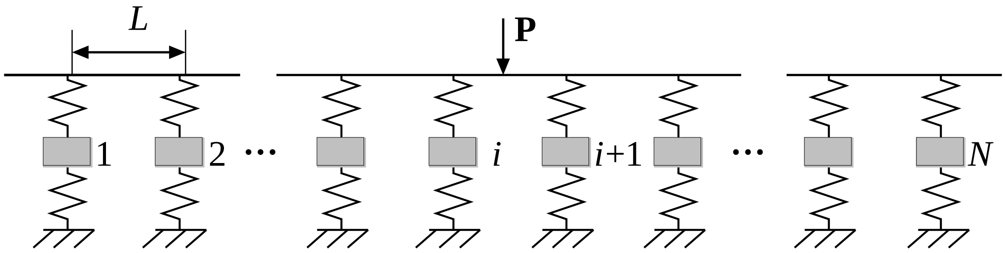

Fig. 2 Track model subjected to a harmonic loading

3. PERTURBATION ANALYSIS OF RESONANT BEHAVIOR

Let us consider a railway track subjected to a harmonic load as shown in Fig. 2. This loading condition can exert a dominant resonance response corresponding to the pinned-pinned mode which is characterized by a motion having nodes at each rail support. This standing wave mode is important in the context of vibration and noise reduction, because it can be excited prominently due to running wheels. Therefore, in this study we exclusively focus on this resonant response.

Steady state amplitude of the rail deflection will depend on the loading frequency and each sleeper position. Its value can thus be regarded as a function of these variables. In this case, the amplitude can be expanded in terms of the perturbations from the dominant resonant frequency f_0 and the regular spacing, i.e.

$$A(\boldsymbol{\varepsilon},\beta) = A_0 + \sum_i A_i\varepsilon_i + \frac{1}{2}\sum_i \sum_j A_{ij}\varepsilon_i\varepsilon_j + \cdots + \left(\dot{A}_0 + \sum_i \dot{A}_i\varepsilon_i + \frac{1}{2}\sum_i \sum_j \dot{A}_{ij}\varepsilon_i\varepsilon_j + \cdots\right)\beta + \frac{1}{2}\left(\ddot{A}_0 + \cdots\right)\beta^2 + \cdots \tag{11}$$

where A is the amplitude at the loading point, $\boldsymbol{\varepsilon} := (\varepsilon_1, \cdots \varepsilon_N)$, A_0 is the resonant amplitude of the unperturbed track at f_0, and $A_i, A_{ij}, \dot{A}_i, \ddot{A}_{ij}$ are coefficients. $\beta = f - f_0$ is the deviation in the loading frequency f from f_0. Since the resonant amplitude $A_0 = A(0,0)$ is an extreme value with respect to the frequency, the following condition is satisfied,

$$\frac{\partial A}{\partial \beta}(0,0) = \dot{A}_0 = 0. \tag{12}$$

From Eqs.(11) and (12), the derivative of A with respect to β can be approximated as

$$\frac{\partial A}{\partial \beta} \approx \left(\sum_i \dot{A}_i\varepsilon_i + \frac{1}{2}\sum_i \sum_j \dot{A}_{ij}\varepsilon_i\varepsilon_j\right) + \ddot{A}_0\beta. \tag{13}$$

Imposing the resonant condition $\partial A/\partial\beta = 0$ on Eq.(13), we can obtain the resonant frequency:

$$\beta = -\frac{1}{\ddot{A}_0}\left(\sum_i \dot{A}_i \varepsilon_i + \frac{1}{2}\sum_i \sum_j \dot{A}_{ij}\varepsilon_i\varepsilon_j\right). \tag{14}$$

Abe et al. (2011) found that the first-order term on the right-hand side of Eq.(14) is negligible. Consequently, the resonant frequency *β* can be expressed as

$$\beta = -\frac{1}{2\ddot{A}_0}\sum_i \sum_j \dot{A}_{ij}\varepsilon_i\varepsilon_j. \tag{15}$$

Substitution of Eq.(15) into (11) results in the following equation:

$$A = A_0 + \sum_i A_i\varepsilon_i + \frac{1}{2}\sum_i \sum_j A_{ij}\varepsilon_i\varepsilon_j - \frac{1}{4\ddot{A}_0}\sum_{ijkl} \dot{A}_{ij}\dot{A}_{kl}\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_l + \cdots. \tag{16}$$

From this equation, it can be seen that the dependence of the resonant frequency on the sleeper location contributes to the fourth-order or higher. The second-order approximation of the resonant amplitude is given by

$$A = A_0 + \sum_i A_i\varepsilon_i + \frac{1}{2}\sum_i \sum_j A_{ij}\varepsilon_i\varepsilon_j. \tag{17}$$

Eq.(17) leads to the mean value and the variance of *A* approximated by the second-order moment as

$$\begin{aligned} E(A) &= A_0 + \frac{1}{2}\sigma_\varepsilon^2 \sum_i A_{ii}, \\ Var(A) &= \sigma_\varepsilon^2 \sum_i A_i^2. \end{aligned} \tag{18}$$

Notice that the variance by the second-order moment in Eq.(18) is identical with that obtained by the first-order perturbation. The variance calculated from Eq.(17) can originally contain forth-order terms. The availability of such higher-order evaluation will be discussed in the next section.

4. NUMERICAL EXAMPLES

4.1 Analytical conditions

In this section we investigate the influence of the randomness in sleeper position on the pinned-pinned resonance mode. We examine whether there is any relationship between the resonant response amplitude and the deviation in the sleeper position. For this purpose, we analyze the change in the resonant response amplitude at the loading position due to the deviation of location introduced into a certain sleeper.

The analytical conditions are illustrated in Fig. 3. A track consisting of a JIS 50kgN rail and sleepers of 100kg per one rail is considered. The values of pad stiffness are k_r =110MN/m and k_s =30MN/m, where k_r and k_s are spring constants of rail and sleeper pads. In the analysis the rail is modeled by a Timoshenko beam and the sleeper is represented by a mass. The rail is supported by 360 sleepers and the standard sleeper spacing is 60cm (*L*=60cm), i.e. the total length is 216m. Notice that, if some adequate damping exists, this track length will be sufficient to suppress the waves reflecting from the rail ends. In order to introduce the damping in the track, the pads are represented by complex stiffness as

$$\begin{aligned} k'_R &= k_R(1+i\omega g), \\ k'_S &= k_S(1+i\omega g), \end{aligned} \tag{19}$$

where $i=\sqrt{-1}$, *ω* is the angular frequency and *g* is a damping coefficient. In the following analyses, *g*=0.01(s/rad). A harmonic loading is located at the mid-span between the 180th and 181st sleepers. The influence of sleeper spacing is discussed based on the resonance deflection at the loading position.

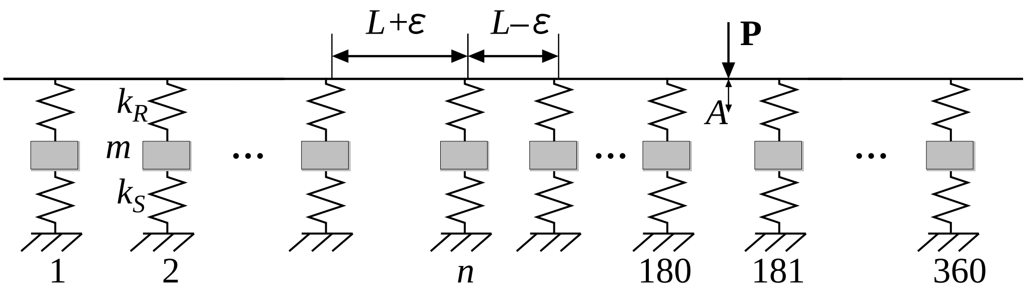

Fig. 3 Track model that *n*th sleeper location is shifted

4.2 Relationship between sleeper position and response amplitude

Fig. 4 shows the relationship between the deviation in the position of a sleeper from the periodicity and the resonant response amplitude at the loading point. In the figure the resonant response amplitude *A* is shown as a function of the deviation ε_n in the *n*th (*n*=181, 220, 260) sleeper location. Needless to say, this behavior is symmetry with respect to the sleeper number, e.g. the influence of the 180th sleeper can be identified by that of the 181st sleeper which locates on the opposite side of the load. The figure shows that, in a region of comparatively small deviation (-0.01m< ε_n <0.01m), the resonant response amplitude increases and shows a convex downward curve. This result implies that the vibration reaction can be larger than that of the resonance mode by introducing a small disturbance in the periodicity. The same phenomenon has been found by Duclos and Clément (2004) for water wave transmission in an array of cylindrical piles. On the other hand, the resonant response amplitude reduces at larger deviation ($|\varepsilon|$ > 0.01m). The influence of the deviation in a sleeper reduces with increasing distance between the loading point and the sleeper. In particular, the resonant behavior is very sensitive to the deviation of adjacent sleepers.

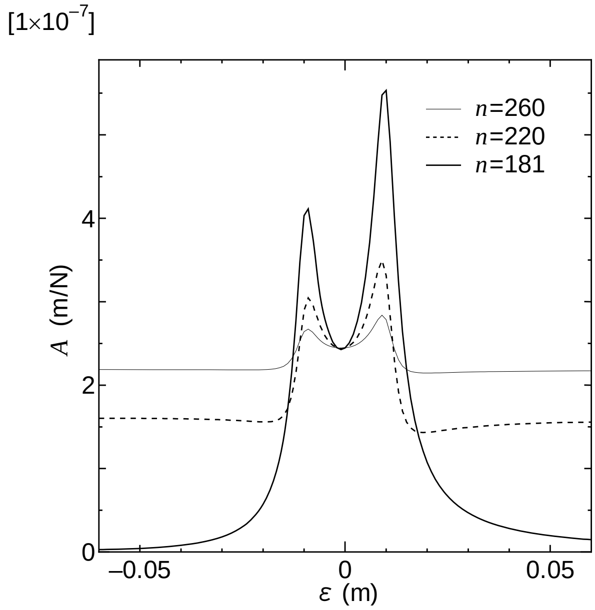

Fig. 4 Relation between ε and *A*

4.3 Evaluation of stochastic behavior

To examine the availability of the approximation Eq.(18), we performed numerical experiments for the track that the disorder in the sleeper spacing is governed by the Gaussian distribution characterized by Eq.(10). The track modeling is the same as shown in *4.1*.

Fig. 5 and Fig. 6 are showing the relationships of σ_ε^2 -*E*(*A*) and σ_ε^2 -*Var*(*A*), respectively. The expectation and the variance of *A* are evaluated by 1000 random tracks generated for each σ_ε^2. From these figures it is confirmed that both *E*(*A*) and *Var*(*A*) increase temporarily with increasing σ_ε^2 but decrease thereafter. Since, in general, the value of σ_ε^2 in the actual railway track is far larger than the range shown in the figure, practically the mean value of the amplitude will be reduced due to the randomness. Although the variance of the response is sensitive to the degree of scatter in the sleeper location as well as that of the mean value, it becomes less sensitive to the deviation at larger σ_ε^2. From this fact, it can be expected that, in many cases, the resonant amplitude decreases with increasing disturbance in the sleeper location, while it may have rather large values in some cases.

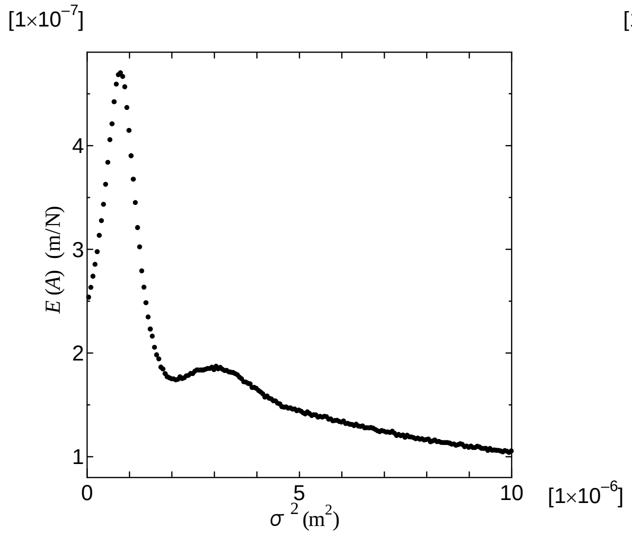

Fig. 5 Relation between σ_ε^2 and *E*(*A*)

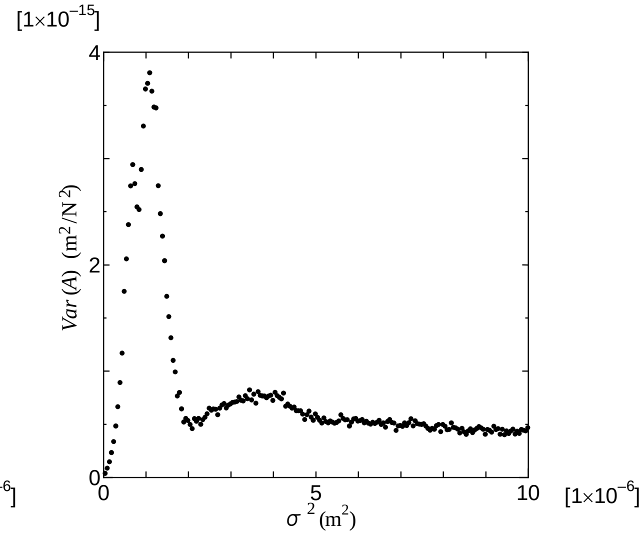

Fig. 6 Relation between σ_ε^2 *and Var*(*A*)

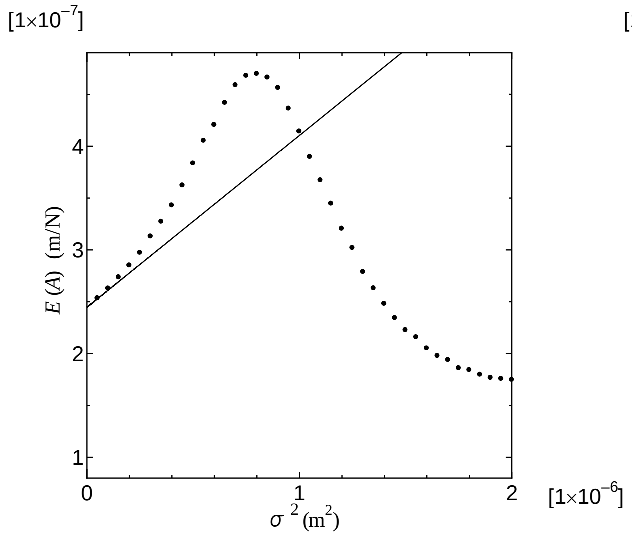

Fig.7 Approximation of relation between σ_ε^2 and $E(A)$

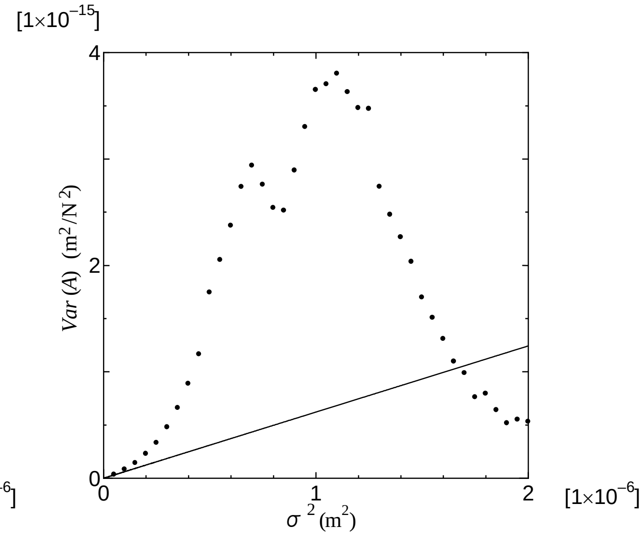

Fig.8 Approximation of relation between σ_ε^2 and $Var(A)$

The approximations of *E*(*A*) and *Var*(*A*) by Eq.(18) are shown by solid lines in Fig. 7 and Fig. 8. In these figures the experimental results shown in Figs.5 and 6 are also plotted. It can be observed that the error of approximation becomes large with increasing σ_ε^2. The applicable range of the second-order perturbation is restricted to $\sigma_\varepsilon^2 < 2\times10^{-7}$ (m^2). This value is very smaller than the variance in the actual railway tracks. Moreover, even if the forth-order terms are considered, it contributes only to the quadratic components in Figs.7 and 8. Therefore, it is obvious that the forth-order approximation is invalid for the evaluation of *E*(*A*) and *Var*(*A*) at a stochastic level which will be observed in a track. Consequently, application of the perturbation method to the evaluation of statistical parameters of resonant amplitude will be difficult.

5. CONCLUSIONS

In this paper, we investigated the influence of random sleeper spacing on the resonant behavior of a railway track. Through numerical analyses, it was found that the resonance response is amplified at a comparatively small deviation in the sleeper position, while it will decrease for rather a large deviation. Also, we have discussed the availability of the perturbation method for the evaluation of the statistical descriptors. Although the perturbation method can approximate the statistical parameters in theoretically, the applicable range is limited to very small deviation. Therefore, it is not practical to evaluate the statistical values by the perturbation approach.

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REFERENCES

Mead, D.J. (1970), “Free wave propagation in periodically supported infinite beam”, *J. Sound Vib.*, 11(2), 181-197.

Hosking, R.J., Husain, S.A. and Milinazzo, F. (2004), “Natural flexural vibrations of a continuous beam on discrete elastic supports”, *J. Sound Vib.*, 272, 169-185.

Wu, T.X. and Thompson, D.J. (2000), “The influence of random sleeper spacing and ballast stiffness on the vibration behaviour of railway track”, *Acustica,* 86, 313-321.

Heckl, M.A. (1995), “Railway noise-Can random sleeper spacings Help? ”, *Acustica*, 81, 559-564.

Batjargal, S., Abe, K. and Koro, K. (2012), “Sleeper spacing optimization for vibration reduction in rails”, *COSEIK J. Comput. Struct. Eng.*, 25(6), 569-577.

Duclos, G. and Clément, A.H. (2004), “Wave propagation through arrays of unevenly spaced vertical piles”, *Ocean Eng.*, 31, 1655-1668.

Li F.-M. and Wang, Y.-S. (2005), “Study on wave localization in disordered periodic layered piezoelectric composite structures”, *I. J. Solids Struct.*, 42, 6457-6474.

Nordborg, A. (1998), “Parametrically excited rail/wheel vibrations due to track-support irregularities”, *Acustica*, 84, 854-859.

Oscarsson, J. (2002), “Dynamic train-track interaction: Variability attributable to scatter in the track properties”, *Vehicle Sys. Dyn.*, 37(1), 59-79.

Abe, K., Shimizu, S., Aikawa, A. and Koro, K. (2011), “Theoretical study on a measuring method of rail axial stress via vibration modes of periodic track”, WCRR2011.